PROPERTIES OF THE MEASURABLE CALCULUS

(14) $f \mapsto \tilde{f}(T)$ is a linear mapping $\mathcal{L}^\infty(E_T) \to \mathcal{L}(H)$

⌈ obvious ⌋

(15) $\tilde{\bar{f}}(T) = \tilde{f}(T)^*$, $f \in \mathcal{L}^\infty(E_T)$

⌈ Let $x \in H$ be arbitrary. Then

$$\langle \tilde{\bar{f}}(T)x, x\rangle = \int_{\sigma(T)} \bar{f}\, dE_{x,x}$$

$$\langle \tilde{f}(T)^* x, x\rangle = \langle x, \tilde{f}(T)x\rangle = \overline{\langle \tilde{f}(T)x, x\rangle} =$$

$$= \overline{\int_{\sigma(T)} f\, dE_{x,x}} \underset{E_{x,x} \geq 0}{=} \int_{\sigma(T)} \bar{f}\, dE_{x,x} = \langle \tilde{\bar{f}}(T)x, x\rangle$$

So, $\tilde{f}(T)^* = \tilde{\bar{f}}(T)$ by Proposition 4 (c) ⌋

(16) $\widetilde{fg}(T) = \tilde{f}(T)\tilde{g}(T)$, $f, g \in \mathcal{L}^\infty(E_T)$

⌈ • we know it works if f, g are cts

• Suppose g is cts.

Let $x, y \in H$

Let (f_n) be a uniformly bdd sequence of cts functions s.t. $f_n \to f$ $|E_{\tilde{g}(T)x,y}| + |E_{x,y}|$-a.e.

Then $\langle \tilde{f}(T)\tilde{g}(T)x, y\rangle = \int_{\sigma(T)} f\, dE_{\tilde{g}(T)x,y} =$

$$\overset{\square}{=} \lim_{n\to\infty} \int_{\sigma(T)} f_n\, dE_{\tilde{g}(T)x,y} = \lim_{n\to\infty} \langle \tilde{f_n}(T)\tilde{g}(T)x, y\rangle =$$

the val. ch to for cts functions

$$= \lim_{n\to\infty} \langle \widetilde{f_n g}(T)x, y\rangle = \lim_{n\to\infty} \int f_n g\, dE_{x,y} \overset{\square}{=} \int fg\, dE_{x,y} = \langle \widetilde{fg}(T)x, y\rangle$$

$\square$ = Lebesgue domin. conv. thm

- $f, g \in \mathcal{L}^\infty(E_T)$ general
  Let $x, y \in H$
  Let (g_n) be a unif. bdd sequence of cts functions s.t.

$$g_n \to g \qquad |E_{x,y}| + |E_{x, \tilde{f}(T)^* y}| - \text{a.e.}$$

$$\langle \tilde{f}(T)\tilde{g}(T)x, y\rangle = \langle \tilde{g}(T)x, \tilde{f}(T)^* y\rangle = \int_{\sigma(T)} g \, dE_{x, \tilde{f}(T)^* y} =$$

Lebesgue dom. conv. thm

$$= \lim_{n\to\infty} \int_{\sigma(T)} g_n \, dE_{x, \tilde{f}(T)^* y} = \lim_{n\to\infty} \langle \tilde{g_n}(T)x, \tilde{f}(T)^* y\rangle =$$

$$= \lim_{n\to\infty} \langle \tilde{f}(T)\tilde{g_n}(T)x, y\rangle = \lim_{n\to\infty} \langle \widetilde{fg_n}(T)x, y\rangle =$$

the previous case

$$= \lim_{n\to\infty} \int_{\sigma(T)} fg_n \, dE_{x,y} = \int_{\sigma(T)} fg \, dE_{x,y} = \langle \widetilde{fg}(T)x, y\rangle$$

Lebesgue domin. conv. thm. $\lrcorner$

(17) Summarizing (14), (15), (16): $f \mapsto \tilde{f}(T)$ is a $*$-homomorphism

$$\mathcal{L}^\infty(E_T) \to \mathcal{L}(H)$$

In part.: f real-valued (except on a set from $\mathcal{N}$) $\Rightarrow \tilde{f}(T)$ self-adjoint

(18) $f \geq 0 \Rightarrow \tilde{f}(T) \geq 0$, Moreover $\tilde{f}(T) = 0 \Leftrightarrow f = 0$)

- $f \geq 0 \Rightarrow \langle \tilde{f}(T)x, x\rangle = \int f \, dE_{x,x} \geq 0$ as $E_{xx} \geq 0$

$\tilde{f}(T) = 0 \Rightarrow \forall x \; \langle \tilde{f}(T)x, x\rangle = 0 \Rightarrow \forall x \int f \, dE_{xx} = 0$

As $f \geq 0$, we deduce $f = 0$ $E_{x,x}$ - a.e.

Hence $f = 0$ except on a set from $\mathcal{N}$

(19) $\tilde{f}(T) = 0 \Rightarrow f = 0$ except on a set from $\mathcal{N}$

$\ulcorner$ $f \geq 0 \Rightarrow$ by (18)

f real-valued $\Rightarrow f' = f^+ - f^-$, $f^+, f^- \in L^\infty(E_T)$

Then $\tilde{f}(T) = \tilde{f^+}(T) - \tilde{f^-}(T)$, thus $\tilde{f^+}(T) = \tilde{f^-}(T)$.

So, $\widetilde{(f^+)^2}T \overset{(16)}{=} \tilde{f^+}(T)\tilde{f^+}(T) = \tilde{f^+}(T)\tilde{f^-}(T) =$

$\overset{(16)}{=} \widetilde{f^+f^-}(T) = \tilde{0}(T) = 0$

Thus $f^+ = 0$ except a set from $\mathcal{N}$ and the same holds for f^-, thus $f = 0$ except a set from $\mathcal{N}$

f complex $\Rightarrow \tilde{f}(T) = \underbrace{\widetilde{\operatorname{Re} f}(T)}_{\text{self-adjoint}} + i\,\widetilde{\operatorname{Im} f}(T)$

$\Rightarrow \widetilde{\operatorname{Re} f}(T) = 0$ & $\widetilde{\operatorname{Im} f}(T) = 0$

$\Rightarrow \operatorname{Re} f = 0$, $\operatorname{Im} f = 0$ except on a set from $\mathcal{N}$

$\Rightarrow f = 0$ except on a set from $\mathcal{N}$. $\lrcorner$

(20) So, $f \mapsto \tilde{f}(T)$ is a $*$-isomorphism, so it is an isometry.

<u>In particular:</u>
- $\tilde{f}(T)$ is always a normal operator
- $\tilde{f}(T)$ is self-adjoint $\Leftrightarrow f$ real valued (except a set from $\mathcal{N}$)
- $\sigma(\tilde{f}(T)) = \sigma(\tilde{f}) = \operatorname{ess\,rng}(f)$

$= \{\lambda \in \mathbb{C};\ \forall r > 0\ f^{-1}(U(\lambda, r)) \notin \mathcal{N}\}$

(21) $(f_n) \subset \mathcal{L}^\infty(E_T)$, $f_n \to f$ pointwise except on a set from $\mathcal{N}$
(f_n) unif. bdd

$$\Rightarrow f \in \mathcal{L}^\infty(E_T), \quad \langle \tilde{f_n}(T)x, y\rangle \to \langle \tilde{f}(T)x, y\rangle$$

[use definitions and Lebesgue dominated conv. thm]

(22) $f \in \mathcal{L}^\infty(E_T)$, $g \in \mathcal{C}(\sigma(\tilde{f}(T))) = \mathcal{C}(\text{ess ran} f)$

$$\Rightarrow \widetilde{g\circ f}(T) = \tilde{g}(\tilde{f}(T))$$

Γ $A := \{ g \in \mathcal{C}(\sigma(\tilde{f}(T))) ; \widetilde{g\circ f}(T) = \tilde{g}(\tilde{f}(T)) \}$

- A is linear
- $1 \in A$, $id \in A$
- $g \in A \Rightarrow \bar{g} \in A$ (as $\bar{g}\circ f = \overline{g\circ f}$)
- $g_1, g_2 \in A \Rightarrow g_1 g_2 \in A$
  Γ $\widetilde{(g_1 g_2)\circ f}(T) = \widetilde{(g_1\circ f)\cdot(g_2\circ f)}(T) = \widetilde{g_1\circ f}(T)\,\widetilde{g_2\circ f}(T)$
  $= \tilde{g_1}(\tilde{f}(T))\,\tilde{g_2}(\tilde{f}(T)) = \widetilde{g_1 g_2}(\tilde{f}(T))$ $\lrcorner\lrcorner$

- A is norm-closed

So, by Stone-Weierstrass thm we get $A = \mathcal{C}(\sigma(\tilde{f}(T)))$ $\lrcorner$

(23) $ST = TS \Rightarrow S\tilde{f}(T) = \tilde{f}(T)S$

Γ • we know it works if f is cts
• f general: Fix $x, y \in H$. Find (f_n) unif. bdd sequence of cts functions s.t. $f_n \to f$ $|E_{Sx,y}| + |E_{x,S^*y}|$ - a.e.

Then $\langle S\tilde{f}(T)x, y\rangle = \langle \tilde{f}(T)x, S^*y\rangle = \int f \, dE_{x,S^*y} = \lim_n \int f_n \, dE_{x,S^*y} =$

$= \lim_n \langle \tilde{f_n}(T)x, S^*y\rangle = \lim_n \langle S\tilde{f_n}(T)x, y\rangle = \lim_n \langle \tilde{f_n}(T)Sx, y\rangle =$

$= \lim_n \int f_n \, dE_{Sx,y} = \int f \, dE_{Sx,y} = \langle \tilde{f}(T)Sx, y\rangle$ $\lrcorner$