Proposition IX.6 H .. a Hilbert space, $P \in \mathcal{L}(H)$ a projection, i.e. $P^2 = P$

(ii) $\Rightarrow$ (iii) P self-adjoint $\Rightarrow$ P normal
[trivial]

(iii) $\Rightarrow$ (i) P normal $\Rightarrow$ P orthogonal (i.e. $\ker P \perp R(P)$)

[by Prop. 5 (a)]

(i) $\Rightarrow$ (iv) P orthogonal $\Rightarrow$ $\langle Px, x\rangle = \|Px\|^2$, $x \in H$

$$\left[\ \langle Px, x\rangle = \langle Px, Px\rangle + \langle \underbrace{Px}_{\in R(P)}, \underbrace{x - Px}_{\in \ker P}\rangle = \langle Px, Px\rangle\ \right]$$

(iv) $\Rightarrow$ (v) $\forall x \in H\ \langle Px, x\rangle = \|Px\|^2$ $\Rightarrow$ $\forall x \in H\ \langle Px, x\rangle \geq 0$

[OBVIOUS]

(v) $\Rightarrow$ (ii) $\forall x \in H\ \langle Px, x\rangle \geq 0$ $\Rightarrow$ $P^* = P$

$$\left[\ \forall x \in H\ \langle P^* x, x\rangle = \langle x, Px\rangle = \overline{\langle Px, x\rangle} = \langle Px, x\rangle \underset{\text{Prop 4(c)}}{\Rightarrow} P^* = P\ \right]$$

(iv) $\Rightarrow$ (vi) $\forall x \in H$: $\langle Px, x\rangle = \|Px\|^2 \Rightarrow \|P\| \leq 1$

$$\left[\ x \in B_H \Rightarrow \|Px\|^2 = \langle Px, x\rangle \leq \|Px\| \cdot \|x\| \leq \|P\|\right.$$

$$\left.\text{So, } \|P\|^2 \leq \|P\|. \text{ Hence } \|P\| \leq 1\ \right]$$

(vi) $\Rightarrow$ (i) $\quad \|P\| \le 1 \Rightarrow P$ orthogonal

• $(\operatorname{Ker} P)^{\perp} \subset R(P)$

⌈ $x \in (\operatorname{Ker} P)^{\perp}$. Then $x - Px \in \operatorname{Ker} P$, so

~~$\langle x$~~ $\langle x - Px, x\rangle = 0$, hence $\langle x, x\rangle = \langle Px, x\rangle$

Thus $\|x\|^2 = \langle Px, x\rangle \le \|Px\| \cdot \|x\| \le \|x\|^2$

So, we have equalities. Thus $\|Px\| = \|x\|$ and Px is a multiple of x

$Px = \alpha x$. But $\langle x, x\rangle = \langle Px, x\rangle = \langle \alpha x, x\rangle = \alpha \langle x, x\rangle$, hence necessarily $\alpha = 1$ (or $x = 0$, but this is a trivial case)

Hence $Px = x$, so $x \in R(P)$. ⌋

• $R(P) \subset (\operatorname{Ker} P)^{\perp}$

⌈ $x \in R(P) \Rightarrow x = x_1 + x_2$, $x_1 \in \operatorname{Ker} P$, $x_2 \in (\operatorname{Ker} P)^{\perp}$

By above $x_2 \in R(P)$.

Thus $x = Px = Px_1 + Px_2 = Px_2 = x_2 \in (\operatorname{Ker} P)^{\perp}$ ⌋

Moreover, if P, Q are OG projections, then $R(P) \perp R(Q) \Leftrightarrow PQ = 0$

⌈ $R(P) \perp R(Q) \Leftrightarrow R(Q) \subset R(P)^{\perp} = \operatorname{Ker} P \Leftrightarrow PQ = 0$ ⌋