

Proposition 1X.5

H Hilbert space, $T \in \mathcal{L}(H)$

T normal $\Leftrightarrow \forall x \in H : \|Tx\| = \|T^*x\|$ (□)

$$\|Tx\| = \|T^*x\| \Leftrightarrow \langle Tx, Tx \rangle = \langle T^*x, T^*x \rangle$$

$$\| \langle T^*Tx, x \rangle \| \qquad \| \langle TT^*x, x \rangle \|$$

So $\forall x \in H \|Tx\| = \|T^*x\| \Leftrightarrow \forall x \in H : \langle T^*Tx, x \rangle = \langle TT^*x, x \rangle$

\Uparrow Prop. 4(c)

$T^*T = TT^*$

Next suppose that T is normal

(a) $\text{Ker } T = \text{Ker } T^* = \mathcal{R}(T)^\perp$

$x \in \text{Ker } T \Leftrightarrow Tx = 0 \Leftrightarrow \|Tx\| = 0 \Leftrightarrow \|T^*x\| = 0 \Leftrightarrow T^*x = 0 \Leftrightarrow x \in \text{Ker } T^*$

So, $\text{Ker } T = \text{Ker } T^*$

Further, $\text{Ker } T^* = \mathcal{R}(T)^\perp$ for any operator T (not only normal)

\subset : $x \in \text{Ker } T^*, y \in \mathcal{R}(T)$. Fix $z \in H$ s.t. $y = Tz$

$\langle x, y \rangle = \langle x, Tz \rangle = \langle T^*x, z \rangle = \langle 0, z \rangle = 0$

So $x \in \mathcal{R}(T)^\perp$

\supset : $x \in \mathcal{R}(T)^\perp \Rightarrow \forall y \in H : \langle x, Ty \rangle = 0$

$\| \langle T^*x, y \rangle \|$

$\Leftrightarrow T^*x = 0 \Rightarrow x \in \text{Ker } T^*$

(b) $\overline{R(T)} = H \Leftrightarrow T$ is one-to-one. Hence $\sigma_{\mathcal{R}}(T) = \emptyset$, $\sigma(T) = \sigma_{\text{ap}}(T)$

$\overline{R(T)} = H \Leftrightarrow R(T)^\perp = \{0\} \stackrel{(a)}{\Leftrightarrow} \ker T = \{0\} \Leftrightarrow T$ is one-to-one

Further, $\lambda \in \mathbb{C} \Rightarrow \lambda I - T$ is ~~normal~~ normal

So $(\lambda I - T)$ is one-to-one $\Leftrightarrow R(\lambda I - T)$ is dense.

Thus $\sigma_{\mathcal{R}}(T) = \emptyset$ and, by Prop. 2(c), $\sigma(T) = \sigma_{\text{ap}}(T)$

(c) $\lambda \in \mathbb{C}$, $x \in H$. Then $Tx = \lambda x \Leftrightarrow T^*x = \bar{\lambda}x$

In particular $\sigma_{\mathcal{P}}(T^*) = \{\bar{\lambda}; \lambda \in \sigma_{\mathcal{P}}(T)\}$

$\lambda \in \mathbb{C} \Rightarrow \lambda I - T$ is normal, $(\lambda I - T)^* = \bar{\lambda}I - T^*$

Thus $\ker(\lambda I - T) = \ker(\bar{\lambda}I - T^*)$ by (a)

(d) $\lambda_1, \lambda_2 \in \sigma_{\mathcal{P}}(T)$, $\lambda_1 \neq \lambda_2 \Rightarrow \ker(\lambda_1 I - T) \perp \ker(\lambda_2 I - T)$

$x \in \ker(\lambda_1 I - T)$, $y \in \ker(\lambda_2 I - T)$

Then $(\lambda_1 - \lambda_2) \langle x, y \rangle = \langle \lambda_1 x, y \rangle - \langle x, \lambda_2 y \rangle =$

$\stackrel{(c)}{=} \langle Tx, y \rangle - \langle x, T^*y \rangle = \langle Tx, y \rangle - \langle T^*x, y \rangle = 0$

so $\langle x, y \rangle = 0$.