Let A be a C^*-algebra and $a \in A$

(a) $a^* = a \Rightarrow \sigma(a) \subset \mathbb{R}$

$\ulcorner$ Recall: $\lambda \in \sigma(x) \Rightarrow |\lambda| \leq \|x\|$

If $\alpha + i\beta \in \sigma(a)$ $(\alpha, \beta \in \mathbb{R}, \beta \neq 0)$, then

$\forall r \in \mathbb{R}$: $\alpha + i\beta + ir \in \sigma(a + ire)$, so

$$\underbrace{|\alpha + i\beta + ir|^2}_{\substack{\| \\ \alpha^2 + (\beta+r)^2 \\ \| \\ \alpha^2+\beta^2+2\beta r + r^2}} \leq \underbrace{\|a + ire\|^2}_{\substack{\| \\ \|(a^* - ire)(a+ire)\| =}}$$

$$= \|a^*a + r^2 e\| \leq \|a^*a\| + r^2$$

So, $2\beta r \leq \|a\|^2 - \alpha^2 - \beta^2$ for all $r \in \mathbb{R}$.

It follows $\beta = 0$. $\lrcorner$

(b) A unital, $a^* = a^{-1} \Rightarrow \sigma(a) \subset \mathbb{T} = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$

$\ulcorner$ $a^* = a^{-1} \Rightarrow a^*a = aa^* = e$.

Hence $\|a\|^2 = \|a^*a\| = \|e\| = 1$, similarly $\|a^*\| = 1$

So, $\|a\| = \|a^*\| = 1$

Thus $\forall \lambda \in \sigma(a)$: $|\lambda| \leq 1$

and $\forall \lambda \in \sigma(a^*) = \sigma(a^{-1})$: $|\lambda| \leq 1$

Observe that $\sigma(a^{-1}) = \{\frac{1}{\lambda}; \lambda \in \sigma(a)\}$ whenever a is invertible

$\ulcorner$ Indeed, then $\lambda e - a = \frac{1}{\lambda} a (a^{-1} - \frac{e}{\lambda})$ for $\lambda \neq 0$

and, since a is invertible, we deduce

$\lambda e - a \in G(A) \Leftrightarrow a^{-1} - \frac{e}{\lambda} \in G(A)$. $\lrcorner$

So, necessarily $\sigma(a) \subset \{\lambda; |\lambda| = 1\}$

$\lrcorner$

(c) $h \in \Delta(A) \Rightarrow h$ is a $*$-homomorphism

Γ • $a^* = a \Rightarrow \sigma(a) \subset \mathbb{R} \overset{\text{Prop 21}}{\Rightarrow} h(a) \in \mathbb{R}$

• a general $\Rightarrow$ $a = b + ic$, b, c selfadjoint

Then $h(a^*) = h(b - ic) = h(b) - ih(c) = \overline{h(b) + ih(c)}$

$= \overline{h(b+ic)} = \overline{h(a)}$. $\lrcorner$