

Theorem

Let A be a Banach algebra and $a \in A$. Then

$$(a) \quad r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

$$(b) \quad A \text{ unital} \Rightarrow R(\lambda, a) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^n} \quad \text{for } |\lambda| > r(a),$$

where the series converges absolutely.

Proof • Also in (a) w.l.o.g. A is unital. So, suppose A is unital.

$$\bullet \quad \lambda \in \sigma(a) \stackrel{L11}{\Rightarrow} \forall n \in \mathbb{N} \quad \lambda^n \in \sigma(a^n), \text{ hence}$$
$$|\lambda^n| \leq r(a^n) \leq \|a^n\|$$

\uparrow Prop. 8 (v)

$$\text{So, } |\lambda| \leq \|a^n\|^{1/n}$$

$$\text{Thus, } r(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}$$

• Fix $\varphi \in A^*$. By Prop. 8 (v) we have

$$\varphi(R(\lambda, a)) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\lambda^{n+1}} \quad \text{for } |\lambda| > \|a\|$$

Moreover, $\lambda \mapsto \varphi(R(\lambda, a))$ is holomorphic on $\{\lambda; |\lambda| > r(a)\}$ by Prop. 8 (iv)

Thus, by uniqueness of Laurent series,

$$\varphi(R(\lambda, a)) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{\lambda^{n+1}} \quad \text{for } |\lambda| > r(a) \quad (*)$$

Therefore, $\forall \lambda; |\lambda| > r(a) : \frac{\varphi(a^n)}{\lambda^{n+1}} \rightarrow 0$

hence $\left(\frac{\varphi(a^n)}{\lambda^{n+1}}\right)_{n=1}^{\infty}$ is bounded

$\varphi \in A^*$ arbitrary \Rightarrow by uniform boundedness principle:

$$\forall \lambda, |\lambda| > R(\varphi) : \left(\frac{a^n}{\lambda^{n+1}} \right)_{n \in \mathbb{N}} \text{ is bdd}$$

Fix $\lambda, |\lambda| > R(\varphi)$. Then there is $C > 0$ s.t.

$$\left\| \frac{a^n}{\lambda^{n+1}} \right\| \leq C \quad \text{for each } n \in \mathbb{N}$$

$$\text{Hence } \|a^n\| \leq C \cdot |\lambda|^{n+1}$$

$$\|a^n\|^{1/n} \leq C^{1/n} \cdot |\lambda| \cdot |\lambda|^{1/n}$$

$$\text{So, } \limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq \limsup_{n \rightarrow \infty} C^{1/n} |\lambda| |\lambda|^{1/n} = |\lambda|$$

$$\text{So, } \limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq R(\varphi)$$

[$|\lambda| > R(\varphi)$ was arbitrary]

Thus:

$$\inf_{n \in \mathbb{N}} \|a^n\|^{1/n} \leq \liminf_{n \rightarrow \infty} \|a^n\|^{1/n} \leq \limsup_{n \rightarrow \infty} \|a^n\|^{1/n} \leq R(\varphi) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}$$

Hence (a) follows.

Finally, if $|\lambda| > R(\varphi)$, then

$$\limsup_{n \rightarrow \infty} \left\| \frac{a^n}{\lambda^{n+1}} \right\|^{1/n} = \limsup_{n \rightarrow \infty} \frac{\|a^n\|^{1/n}}{|\lambda| \cdot |\lambda|^{1/n}} \stackrel{(a)}{\leq} \frac{R(\varphi)}{|\lambda|} < 1,$$

hence the series from (b) converges absolutely.

Moreover, its sum is $R(\lambda, \varphi)$ by (4).