

(a) Let A be a Banach algebra with unit e
 If $x \in A$ s.t. $\|x\| < 1$, then $e-x$ is invertible

$$\text{and } (e-x)^{-1} = \sum_{n=0}^{\infty} x^n \quad (= e + x + x^2 + x^3 + \dots),$$

where the series converges absolutely

Proof: $\|x^n\| \leq \|x\|^n$ (as $\|x \cdot y\| \leq \|x\| \cdot \|y\|$)

$$\text{Since } \|x\| < 1, \sum_{n=0}^{\infty} \|x\|^n < \infty, \text{ so } \sum_{n=0}^{\infty} \|x^n\| < \infty.$$

Hence, the series converges absolutely. By completeness of A it converges. Set $y := \sum_{n=0}^{\infty} x^n$

$$\text{Then } y \cdot x = \left(\lim_{N \rightarrow \infty} \left(\sum_{n=0}^N x^n \right) \right) \cdot x = \lim_{N \rightarrow \infty} \sum_{n=0}^N x^{n+1} =$$

as multiplication iscts
 (cf Prop 4(b))

$$= \sum_{n=0}^{\infty} x^{n+1} = y - e$$

$$\text{Similarly } x \cdot y = \sum_{n=0}^{\infty} x^{n+1} = y - e$$

$$\text{So, } \begin{cases} y \cdot (e-x) = y - yx = e \\ (e-x)y = y - xy = e \end{cases} \Rightarrow y = (e-x)^{-1}$$

(b) $x \in G(A)$, $h \in A$ s.t. $\|h\| < \frac{1}{\|x^{-1}\|}$

$$\text{Then } x+h \in G(A), \quad (x+h)^{-1} = x^{-1} \cdot \sum_{n=0}^{\infty} (-1)^n (h \cdot x^{-1})^n$$

$$\text{and } \|(x+h)^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \|h\|}{1 - \|x^{-1}\| \|h\|}$$

Proof: $\|k\| < \frac{1}{\|x^{-1}\|} \Rightarrow \|k \cdot x^{-1}\| \leq \|k\| \cdot \|x^{-1}\| < 1$

So, by (a) we get $e + kx^{-1} \in G(A)$ and

$$(e + kx^{-1})^{-1} = \sum_{n=0}^{\infty} (-1)^n (kx^{-1})^n$$

Further, $x + k = (e + kx^{-1}) \cdot x \in G(A)$
(by Prop. 5(a))

$$\text{and } (x+k)^{-1} = x^{-1} (e + kx^{-1})^{-1} = x^{-1} \cdot \sum_{n=0}^{\infty} (-1)^n (kx^{-1})^n$$

$$\begin{aligned} \text{Moreover, } \|(x+k)^{-1} - x^{-1}\| &= \left\| x^{-1} \cdot \sum_{n=1}^{\infty} (-1)^n (kx^{-1})^n \right\| \leq \\ &\leq \|x^{-1}\| \cdot \left\| \sum_{n=1}^{\infty} (-1)^n (kx^{-1})^n \right\| \leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|(-1)^n (kx^{-1})^n\| \\ &\leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} (\|k\| \cdot \|x^{-1}\|)^n = \|x^{-1}\| \cdot \frac{\|k\| \cdot \|x^{-1}\|}{1 - \|k\| \cdot \|x^{-1}\|} \end{aligned}$$