REMARKS ON THE POINT CHARACTER OF BANACH SPACES AND NON-LINEAR EMBEDDINGS INTO $c_0(\Gamma)$

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This work is dedicated to Gilles Godefroy.

ABSTRACT. We give a brief survey of the results on coarse or uniform embeddings of Banach spaces into $c_0(\Gamma)$ and the point character of Banach spaces. In the process we prove several new results in this direction (for example we determine the point character of the spaces $L_p(\mu)$, $1 \leq p \leq 2$) solving open problems posed by C. Avart, P. Komjáth, and V. Rödl and by G. Godefroy, G. Lancien, and V. Zizler. In particular, we show that $X = L_p(\mu)$, $1 \leq p < \infty$, bi-Lipschitz embeds into $c_0(\Gamma)$ if and only if dens $X < \omega_\omega$.

The Banach space c_0 plays a fundamental role both in the linear and non-linear structural theory of Banach spaces. Indeed, by the famous result of Mordecay Zippin [Z] it is the unique separable Banach space which is linearly complemented in every separable superspace. On the other hand, a celebrated result in non-linear functional analysis, due to Israel Aharoni (Theorem 2 below), claims that every separable metric space admits a bi-Lipschitz embedding into the Banach space c_0. It is apparently unknown if the latter property also admits a converse statement, namely if a Banach space that contains a bi-Lipschitz copy of c_0 also contains a linear copy thereof.

Our present note is focusing on embeddings into the non-separable version of c_0, namely the space $c_0(\Gamma)$. We are going to survey some known results and prove several new theorems. The main body of work in this area is due to Jan Pelant and Vojtěch Rödl, and their coauthors. These researchers were originally motivated by studying the covering properties of metric (or uniform) spaces, and the connection with the uniform embeddings into $c_0(\Gamma)$ was discovered somewhat later by J. Pelant (Theorem 8 below). In fact, for normed linear spaces the existence of uniform embeddings is equivalent to the existence of bi-Lipschitz ones (this was known to J. Pelant) as well as the coarse ones (a result of Andrew Swift). One of our main new contributions in this note is contained in Theorem 13, which implies in particular that the embeddability of a normed linear space into $c_0(\Gamma)$ is equivalent to the embeddability of its unit ball. Theorem 14 summarises known equivalent conditions for a normed linear space to be uniformly embeddable into $c_0(\Gamma)$. In Theorem 19, resp. Corollary 21 we improve the results of [PR] and [HS], showing that density ω_ω and larger is an obstacle for embeddings into $c_0(\Gamma)$ for all normed linear spaces of a non-trivial cotype.

The rest of the paper depends to a large extent on the characterisation of those sets Λ, for which $\ell_1(\Lambda)$ embeds into some $c_0(\Gamma)$, which was obtained in [PR] and [AKR] (Theorems 17 and 18). Namely, such embeddings exist if and only if card $\Lambda < \omega_\omega$. Using the uniform equivalence of unit balls of certain Banach lattices (Theorems 23 and 24 below; in particular spaces of non-trivial cotype with an unconditional basis) together with the fundamental embeddability result and Theorem 14 we prove in Corollary 26 that several classes of Banach spaces of a non-trivial cotype and density less than ω_ω embed into $c_0(\Gamma)$. The result holds true in particular for spaces $L_p(\mu)$, $1 \leq p < \infty$. Finally, in Corollary 29 we apply our results to the theory of approximations of Lipschitz mappings by smooth Lipschitz mapping in case when the range space is an absolute Lipschitz retract (e.g. any separable $C(K)$-space).

Let us now pass to the technical part of our note. Let X be a metric space. By $U(x, r)$, resp. $B(x, r)$, resp. $S(x, r)$ we denote the open ball, resp. closed ball, resp. sphere centred at $x \in X$ with radius $r > 0$. If it is necessary to distinguish the metric space in which the ball is taken we use the notation $U_X(x, r)$ etc. We begin by introducing several important concepts regarding uniformity properties of non-linear mappings between metric spaces and the related non-linear embeddings.

Definition 1. Let (X, ρ) and (Y, σ) be metric spaces and $f : X \to Y$. We say that f is of

- bounded expansion if for every $d > 0$ there exists $K \geq 0$ such that $\sigma\big(f(x), f(y)\big) \leq K$ whenever $x, y \in X$ are such that $\rho(x, y) \leq d$;
- bounded compression if for every $K > 0$ there exists $d > 0$ such that $\sigma\big(f(x), f(y)\big) \geq K$ whenever $x, y \in X$ are such that $\rho(x, y) \geq d$.

The mapping f is called a *coarse embedding* if it is both of bounded expansion and bounded compression. It is called a *uniform embedding* if it is one-to-one and both f and f^{-1} are uniformly continuous. It is called a *bi-Lipschitz embedding* if it is one-to-one and both f and f^{-1} are Lipschitz.

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Note that a coarse embedding may not be an embedding (i.e. one-to-one) and if f is one-to-one, then it is a coarse embedding if and only if both f and f^{-1} are of bounded expansion. Further, the notion of coarse embedding is non-trivial only for unbounded metric spaces.

The theory of Lipschitz mappings between separable Banach spaces, which is based on differentiability concepts, has been extensively developed by many authors (see [BL]). In particular, if X, Y are separable Banach spaces and Y has the RNP, then the existence of a bi-Lipschitz embedding of X into Y implies that X is linearly isomorphic to a subspace of Y, [HM]. However, the space c_0 fails the RNP, so this theory cannot be applied in the situation of embeddings into c_0 (or $c_0(\Gamma)$). Instead, we have a fundamental result due to I. Aharoni:

Theorem 2 ([A])**.** *Every separable metric space admits a bi-Lipschitz embedding into c_0^+ (the non-negative cone of c_0).*

J. Pelant [P3] has found the optimal bi-Lipschitz constant in the above result to be 3 (see also [KL]).

Our focus will be on the non-separable version of the Aharoni theorem, i.e. embeddings (uniform, coarse, or bi-Lipschitz) of Banach spaces into $c_0(\Gamma)$. This problem, in the more general setting of embedding metric, or even uniform spaces, has an interesting history. In our note we will restrict our attention (i.e. we will introduce and treat the relevant concepts) to the metric case.

We will be using a topological approach introduced by J. Pelant, which appears unrelated at a first glance. A covering of a set A is a collection $\mathcal{U}$ of subsets of A such that $A = \bigcup_{U\in\mathcal{U}} U$. A covering $\mathcal{U}$ is called point-finite if for every $x \in A$ the set $\{U \in \mathcal{U};\ x \in U\}$ is finite. A covering $\mathcal{V}$ of A is called a refinement of a covering $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$. By the well-known Stone theorem [St] every metric space X is paracompact, i.e. every open covering of X has a locally finite open refinement (in particular every open covering has a point-finite refinement).

A uniform version of the latter property can be formulated as follows. Let X be a metric space. By $\mathcal{U}(r) = \{U(x,r);\ x \in X\}$ we denote the full uniform covering of X. A covering $\mathcal{U}$ of X is called uniform if there exists an $r > 0$ such $\mathcal{U}(r)$ refines $\mathcal{U}$ (in such case we say that $\mathcal{U}$ is r-uniform). It is called uniformly bounded if there is $R > 0$ such that $\operatorname{diam} U \leq R$ for every $U \in \mathcal{U}$ (in such case we say that $\mathcal{U}$ is R-bounded).

Definition 3 ([PHK])**.** A metric space X is said to have the uniform Stone property if every uniform covering of X has a point-finite uniform refinement.

Definition 4 ([Sw])**.** A metric space X is said to have the coarse Stone property if every uniformly bounded covering of X is a refinement of a point-finite uniformly bounded covering of X.

Arthur H. Stone has asked (see [P2]) if a uniform version of the Stone theorem holds, i.e. if each uniform covering of a metric space has a locally finite uniform refinement (or, equivalently, a point-finite uniform refinement, see e.g. [I]). In fact, the problem was posed in the setting of uniform spaces which is more general; we will not introduce the concept of uniform spaces in our note. The problem was solved in the negative independently by Evgeniy V. Shchepin [Sh] and J. Pelant [P1]. E. Shchepin proved that the Banach space $\ell_\infty(\Gamma)$ fails the uniform Stone property whenever $\operatorname{card}\Gamma > \exp_\omega(\omega)$. The proof in [P1] implies, in the so-called Baumgartner model of ZFC, that for any cardinal $\kappa \geq 2^{\omega_1}$ the Banach space $\ell_\infty(\kappa)$ does not have the uniform Stone property. These results have prompted the definition of the point character of a metric/uniform space ([P1], [P2], [R2], [PR], [P3]):

Definition 5. For a collection $\mathcal{E}$ of subsets of some set its order is defined by

- $\operatorname{ord}\mathcal{E} = n$ if $n = \max\{\operatorname{card}\mathcal{D};\ \mathcal{D} \subset \mathcal{E}, \bigcap\mathcal{D} \neq \emptyset\} \in \mathbb{N}$,
- $\operatorname{ord}\mathcal{E} = \alpha$ if $\alpha = \sup\{(\operatorname{card}\mathcal{D})^+;\ \mathcal{D} \subset \mathcal{E}, \bigcap\mathcal{D} \neq \emptyset\}$ is an infinite cardinal.

Definition 6. Let X be a metric space. Its point character $\operatorname{pc} X$ is the least cardinal α such that every uniform covering of X has a uniform refinement $\mathcal{V}$ with $\operatorname{ord}\mathcal{V} < \alpha$.

It is important to note that the definition of the point character in the literature varies and some authors (usually those interested in combinatorics rather than topology) use the following definition (which we will call the reduced point character):

Definition 7. Let X be a metric space. Its reduced point character $\operatorname{rpc} X$ is the least cardinal α such that every uniform covering of X has a uniform refinement $\mathcal{V}$ with $\operatorname{ord}\mathcal{V} \leq \alpha$.

The difference between pc and rpc is that the cardinal pc can differentiate between certain situations occurring at limit cardinals. For example assume that X is such that $\operatorname{rpc} X = \omega$. Then either for each uniform covering of X there is a uniform refinement with a finite order, but for different coverings this order needs to be arbitrarily high. In this case $\operatorname{pc} X = \omega$. Or there is a uniform covering of X that has no uniform refinement of a finite order (but has a uniform refinement of order ω; note that this refinement is still point-finite). In this case $\operatorname{pc} X = \omega_1$. On the other hand if $\operatorname{rpc} X$ is a successor cardinal, then $\operatorname{pc} X = (\operatorname{rpc} X)^+$.

Note however that if X is a normed linear space, then thanks to homogeneity the former situation described above cannot happen and so always $\operatorname{pc} X = (\operatorname{rpc} X)^+$. Since we are primarily interested in normed linear spaces rather than general metric spaces, we will use the cardinal rpc, which leads to more canonical formulas.

It is clear that a metric space X has the uniform Stone property if and only if $\operatorname{rpc} X \leq \omega$ (if and only if $\operatorname{pc} X \leq \omega_1$). The following crucial result of J. Pelant provides a link between the embeddings into $c_0(\Gamma)$ and the uniform covering properties.

Theorem 8 ([P3, Corollary 2.4])**.** *A metric space X satisfies* $\operatorname{rpc} X \leq \omega$ *if and only if it admits a uniform embedding into $c_0(\Gamma)$, where* $\operatorname{card}\Gamma \leq \operatorname{dens} X$.

In general the embedding cannot be bi-Lipschitz, [P3, Corollary 4.4].

We begin our investigation by observing several simple facts concerning the point character. It is useful to notice that since the notion of being a refinement is transitive and in normed linear spaces we can use homogeneity, it is possible to reformulate the definitions of point character in the following way:

Fact 9. *Let X be a metric space.*

- $\operatorname{pc} X$ *is the least cardinal α such that for every $r > 0$ the covering $\mathcal{U}(r)$ of X has a uniform refinement $\mathcal{V}$ with* $\operatorname{ord} \mathcal{V} < \alpha$.
- $\operatorname{pc} X$ *is the least cardinal α such that for every $r > 0$ there exists an r-bounded uniform covering $\mathcal{U}$ of X with* $\operatorname{ord} \mathcal{U} < \alpha$.

If X is a normed linear space, then

- $\operatorname{pc} X$ *is the least cardinal α such that for some $r > 0$ the covering $\mathcal{U}(r)$ of X has a uniform refinement $\mathcal{V}$ with* $\operatorname{ord} \mathcal{V} < \alpha$.
- $\operatorname{pc} X$ *is the least cardinal α such that there exists a bounded uniform covering $\mathcal{U}$ of X with* $\operatorname{ord} \mathcal{U} < \alpha$.

Analogous statements hold for the cardinal rpc.

We will use the above reformulations freely without mention.

Fact 10. *Let X, Y be metric spaces.*
(a) If Y is uniformly homeomorphic to X, then $\operatorname{pc} Y = \operatorname{pc} X$ *and* $\operatorname{rpc} Y = \operatorname{rpc} X$.
(b) If $Y \subset X$, then $\operatorname{pc} Y \leq \operatorname{pc} X$ *and* $\operatorname{rpc} Y \leq \operatorname{rpc} X$.

Proof. (a) Let $\Phi\colon (Y, \sigma) \to (X, \rho)$ be a uniform homeomorphism. Let $r > 0$. There is $\delta > 0$ such that $\sigma\bigl(\Phi^{-1}(x), \Phi^{-1}(y)\bigr) \leq r$ whenever $x, y \in X$, $\rho(x, y) \leq \delta$. Let $\mathcal{V}$ be a δ-bounded uniform covering of X with $\operatorname{ord} \mathcal{V} < \operatorname{pc} X$ (resp. $\operatorname{ord} \mathcal{V} \leq \operatorname{rpc} X$). Let $s > 0$ be such that $\mathcal{V}$ is s-uniform. There is $\varepsilon > 0$ such that $\rho\bigl(\Phi(x), \Phi(y)\bigr) < s$ whenever $x, y \in Y$, $\sigma(x, y) < \varepsilon$. Then $U_Y(x, \varepsilon) \subset \Phi^{-1}\bigl(U_X(\Phi(x), s)\bigr)$ for every $x \in Y$ and so $\mathcal{U} = \{\Phi^{-1}(V);\ V \in \mathcal{V}\}$ is an r-bounded ε-uniform covering of Y. Also, $\operatorname{ord} \mathcal{U} = \operatorname{ord} \mathcal{V}$, since Φ is a bijection. Therefore $\operatorname{pc} Y \leq \operatorname{pc} X$, resp. $\operatorname{rpc} Y \leq \operatorname{rpc} X$. The reverse inequalities follow by symmetry.

(b) Let $r > 0$. Let $\mathcal{V}$ be an r-bounded uniform covering of X with $\operatorname{ord} \mathcal{V} < \operatorname{pc} X$, resp. $\operatorname{ord} \mathcal{V} \leq \operatorname{rpc} X$. Let $s > 0$ be such that $\mathcal{V}$ is s-uniform. Put $\mathcal{U} = \{V \cap Y;\ V \in \mathcal{V}\}$. For every $x \in Y$ there is $V \in \mathcal{V}$ such that $U_X(x, s) \subset V$ and so $U_Y(x, s) \subset V \cap Y$. Thus $\mathcal{U}$ is an r-bounded s-uniform covering of Y. Finally, $\operatorname{ord} \mathcal{W} \leq \operatorname{ord} \mathcal{V}$. Hence $\operatorname{pc} Y \leq \operatorname{pc} X$, resp. $\operatorname{rpc} Y \leq \operatorname{rpc} X$. □

Next, let us mention an easy (but loose) estimate of the cardinal rpc, see e.g. [P3, Lemma 1.1]:

Fact 11. *Let X be a metric space. Then* $\operatorname{rpc} X \leq \operatorname{dens} X$.

The topological theory of dimension yields the following result: $\operatorname{rpc} B_{\mathbb{R}^n} = \operatorname{rpc} \mathbb{R}^n = n + 1$ (see e.g. [Sm, Theorem 11] with [I, Theorem V.5]). Combining this with Fact 10 it follows that if X is an infinite-dimensional normed linear space, then $\operatorname{rpc} X \geq \operatorname{rpc} B_X \geq \omega$.

If α is a cardinal, then $\operatorname{cf} \alpha$ denotes the cofinality of α (see [Je, p. 31]). We will make use of the following lemma, which is based on a trick from Mary Ellen Rudin's proof of the Stone paracompactness theorem (cf. also [AKR]).

Lemma 12. *Let $\mathcal{U}$ be an r-uniform covering of a metric space X. Assume that α is an infinite cardinal such that $\mathcal{U} = \bigcup_{\gamma \in \Gamma} \mathcal{U}_\gamma$ with* $\operatorname{ord} \mathcal{U}_\gamma \leq \alpha$ *for each $\gamma \in \Gamma$ and* $\operatorname{card} \Gamma \leq \operatorname{cf} \alpha$. *Then $\mathcal{U}$ has an $\frac{r}{2}$-uniform refinement $\mathcal{V}$ with* $\operatorname{ord} \mathcal{V} \leq \alpha$.

Proof. We may assume without loss of generality that Γ is the ordinal interval $[1, \operatorname{card} \Gamma)$ and that $\mathcal{U}_\gamma$, $\gamma \in \Gamma$, are pairwise disjoint in the sense that each $U \in \mathcal{U}$ belongs to precisely one $\mathcal{U}_\gamma$. On each $\mathcal{U}_\gamma$, $\gamma \in \Gamma$, choose some well-ordering $\leq$. Let $\preceq$ be a lexicographic ordering on $\mathcal{U}$ induced by $(\Gamma, \leq)$ and $(\mathcal{U}_\gamma, \leq)$. Given any $A \subset X$ let us denote $\widetilde{A} = \{x \in X;\ U(x, r) \subset A\}$. For $U \in \mathcal{U}$ set $\widehat{U} = U \setminus \bigcup_{V \in \mathcal{U}, V \prec U} \widetilde{V}$. Then $\mathcal{V} = \{\widehat{U};\ U \in \mathcal{U}\}$ is the desired refinement. Indeed, let $x \in X$. Let $U \subset \mathcal{U}$ be the first one in the ordering $\preceq$ such that $U(x, \frac{r}{2}) \subset U$ (such U exists, since $\mathcal{U}$ is r-uniform). It follows that $U(x, \frac{r}{2}) \cap \widetilde{V} = \emptyset$ for every $V \in \mathcal{U}$, $V \prec U$, since $U(x, \frac{r}{2}) \subset U(z, r)$ for any $z \in U(x, \frac{r}{2})$. Consequently, $U(x, \frac{r}{2}) \subset \widehat{U}$.

To see that $\operatorname{ord} \mathcal{V} \leq \alpha$ let $\mathcal{D} \subset \mathcal{V}$ be such that $\bigcap \mathcal{D} \neq \emptyset$. For $\gamma \in \Gamma$ set $\mathcal{D}_\gamma = \{\widehat{V} \in \mathcal{D};\ V \in \mathcal{U}_\gamma\}$. Then $\mathcal{D} = \bigcup_{\gamma \in \Gamma} \mathcal{D}_\gamma$ and $(\operatorname{card} \mathcal{D}_\gamma)^+ \leq \alpha$, since $\bigcap \mathcal{D}_\gamma \neq \emptyset$ in case that $\mathcal{D}_\gamma \neq \emptyset$. Let $x \in \bigcap \mathcal{D}$ and let $U \in \mathcal{U}$ be the first one in the ordering $\preceq$ such that $U(x, r) \subset U$. It follows that $x \in \widetilde{U}$ and consequently $x \notin \widehat{V}$ for any $V \in \mathcal{U}$, $V \succ U$. Let $\beta \in \Gamma$ be such that $U \in \mathcal{U}_\beta$. Then $\mathcal{D}_\gamma = \emptyset$ for $\gamma > \beta$ and so $\mathcal{D} = \bigcup_{\gamma \leq \beta} \mathcal{D}_\gamma$. Since $\operatorname{card} \mathcal{D}_\gamma < \alpha$ and $\beta < \operatorname{cf} \alpha$, it follows that $\operatorname{card} \mathcal{D} < \alpha$. Therefore $(\operatorname{card} \mathcal{D})^+ \leq \alpha$. □

One of our main tools is the following result:

Theorem 13. *Let X be a normed linear space. If* $\operatorname{rpc} B_X \leq \alpha$, *where α is an infinite regular cardinal, then* $\operatorname{rpc} X \leq \alpha$. *Consequently if* $\operatorname{rpc} B_X$ *is an infinite regular cardinal, then* $\operatorname{rpc} X = \operatorname{rpc} B_X$.

Proof. In this proof every covering of any $A \subset X$ is understood as a covering of the metric space A, i.e. a covering by subsets of A. Let $r > 0$ be such that B_X has a $\frac{1}{2}$-bounded r-uniform covering of order at most α. The crucial step of the proof is the following observation:

(∗) Suppose that $A \subset X$ has a $\frac{1}{2}$-bounded r-uniform covering of order at most α. Then $B = 2A$ also has a $\frac{1}{2}$-bounded r-uniform covering of order at most α.

To see this let $\mathcal{U}$ be a $\frac{1}{2}$-bounded r-uniform covering of A with $\operatorname{ord}\mathcal{U} \leq \alpha$. Then $\mathcal{V} = \{2U;\ U \in \mathcal{U}\}$ is a 1-bounded $2r$-uniform covering of B with $\operatorname{ord}\mathcal{V} \leq \alpha$. Since each $V \in \mathcal{V}$ is a subset of some ball $B_X(w, 1)$, it has a $\frac{1}{2}$-bounded r-uniform covering $\mathcal{W}_V$ with $\operatorname{ord}\mathcal{W}_V \leq \alpha$. Set $\mathcal{W} = \bigcup_{V\in\mathcal{V}} \mathcal{W}_V$. Then $\mathcal{W}$ is clearly $\frac{1}{2}$-bounded. Further, given any $x \in B$ there is $V \in \mathcal{V}$ such that $U_B(x, r) = U_X(x, r) \cap B \subset V$ and there is $W \in \mathcal{W}_V$ such that $U_V(x, r) = U_X(x, r) \cap V \subset W$. Hence $U_B(x, r) = U_V(x, r) \subset W$. It follows that $\mathcal{W}$ is an r-uniform covering of B.

Finally, to see that $\operatorname{ord}\mathcal{W} \leq \alpha$ let $\mathcal{D} \subset \mathcal{W}$ be such that $\bigcap \mathcal{D} \neq \emptyset$. For $V \in \mathcal{V}$ set $\mathcal{D}_V = \{W \in \mathcal{D};\ W \in \mathcal{W}_V\}$. Let $x \in \bigcap \mathcal{D}$ and let $\mathcal{A} = \{V \in \mathcal{V};\ x \in V\}$. Then $\bigcap \mathcal{A} \neq \emptyset$ and hence $\operatorname{card}\mathcal{A} < \alpha = \operatorname{cf}\alpha$. Note that $W \subset V$ whenever $W \in \mathcal{D}_V$ and $V \in \mathcal{V}$. Thus $\mathcal{D}_V = \emptyset$ for $V \in \mathcal{V} \setminus \mathcal{A}$. Hence $\mathcal{D} = \bigcup_{V\in\mathcal{A}} \mathcal{D}_V$. Since $\operatorname{card}\mathcal{D}_V < \alpha$ for $V \in \mathcal{A}$ as $\bigcap \mathcal{D}_V \neq \emptyset$, it follows that $\operatorname{card}\mathcal{D} < \alpha$ and so $(\operatorname{card}\mathcal{D})^+ \leq \alpha$

To finish, using observation $(*)$ inductively starting with $A = B_X$ we conclude that for each $n \in \mathbb{N}$ the ball $B_X(0, 2^n)$ has a $\frac{1}{2}$-bounded r-uniform covering $\mathcal{U}_n$ with $\operatorname{ord}\mathcal{U}_n \leq \alpha$. It follows that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a uniformly bounded uniform covering of X: for each $x \in X$ consider $n \in \mathbb{N}$ satisfying $2^n \geq \|x\| + r$, then $U_X(x, r) \subset B_X(0, 2^n)$ and there is $U \in \mathcal{U}_n$ such that $U_X(x, r) \subset U \subset B_X(0, 2^n)$. Finally, $\mathcal{U}$ has a uniform refinement of order at most α by Lemma 12.

The last statement of the theorem follows from Fact 10(b).

□

The following theorem, which summarises properties equivalent to uniform embedding into $c_0(\Gamma)$, is a combination of results of J. Pelant [P3], A. Swift [Sw], as well as some new ones:

Theorem 14. *Let X be a normed linear space. The following statements are equivalent:*

(i) $\operatorname{rpc} X \leq \omega$.
(ii) X has the uniform Stone property.
(iii) X has the coarse Stone property.
(iv) X admits a coarse embedding into $c_0(\Gamma)$ for some set Γ.
(v) X admits a uniform embedding into $c_0(\Gamma)$ for some set Γ.
(vi) X admits a bi-Lipschitz embedding into $c_0(\Gamma)$ for some set Γ.
(vii) $\operatorname{rpc} B_X \leq \omega$.
(viii) B_X admits a uniform embedding into $c_0(\Gamma)$ for some set Γ.
(ix) B_X admits a bi-Lipschitz embedding into $c_0(\Gamma)$ for some set Γ.

The equivalence with statements (vii)–(ix) is new.

Proof of Theorem 14. (i)⇔(ii) is just the definition. (i)⇔(v) and (vii)⇔(viii) follow from Theorem 8. (vi)⇒(v) is trivial, (v)⇒(vi) is in [P3, Remark 4.6(3)] (cf. also [HJ1, Theorem 3]). (ii)⇔(iii) is in [Sw, Lemma 3.7]. (iv)⇔(v) is in [Sw, Corollary 3.14]. (vi)⇒(ix)⇒(viii) is trivial. (vii)⇒(i) follows from Theorem 13.

□

In light of the above characterisations it is clear that the problem of uniform embedding into $c_0(\Gamma)$ is not just a special result and moreover it has a general answer. However, deciding the embeddability for concrete spaces is not so easy. Let us describe some crucial results in this direction, which were in fact obtained before Theorem 8 was discovered.

V. Rödl [R2] constructed a metric space X with $\operatorname{card} X = \omega_1$ and $\operatorname{rpc} X = \omega_1$.

An interesting problem, posed in [GLZ], is whether ℓ_∞ embeds uniformly into some $c_0(\Gamma)$. In view of Theorem 14 this is equivalent to asking whether $\operatorname{rpc}\ell_\infty \leq \omega$. [Ho, Theorem 5.1] would imply that the answer is negative. Unfortunately the argument of A. Hohti is not correct and the problem seems to be still open. Similarly, the authors in [PR] mention an argument of J. Pelant that $\operatorname{rpc}\ell_\infty > \omega$, but the details have not been published. The best result in this direction are due to J. Pelant. First in [P1], using the Baumgartner model of ZFC, it is shown that $\operatorname{rpc}\ell_\infty(2^{\omega_1}) > \omega$. Later, in [P2, Remark on p. 160], J. Pelant gives an improvement, which seems to be proved using only ZFC:

Theorem 15 ([P2])**.** *If $\beta < \alpha$ are cardinals, where β is regular, then* $\operatorname{rpc}\ell_\infty(\alpha) > \beta$.

In particular, $\operatorname{rpc}\ell_\infty(\omega_1) > \omega$. Considering the model of set theory in which $\mathfrak{c} > \omega_\omega$ and using the well-known structural result that $\ell_1(\mathfrak{c})$ is isometric to a subspace of ℓ_∞ together with Theorem 17 below, we obtain the following observation:

Theorem 16. $\operatorname{rpc}\ell_\infty \geq \omega_\omega$ *if and only if* $\mathfrak{c} > \omega_\omega$.

Proof. ⇒ follows from the fact that $\operatorname{dens}\ell_\infty = \mathfrak{c}$, Fact 11, and the fact that $\operatorname{cf}\mathfrak{c} > \omega$ ([Je, Theorem 5.16]). ⇐ follows from the fact that $\ell_1(\mathfrak{c})$ is isometric to a subspace of ℓ_∞ ([FHHMZ, Exercise 5.34]) and Theorem 17.

□

So the answer to the question is consistently negative, but a ZFC proof is yet to be found.

Another class of Banach spaces for which rpc has been studied are $\ell_p(\Gamma)$, $1 \leq p < \infty$. An early result of V. Rödl [R1] is that $\operatorname{rpc}\ell_1(\Gamma)$ can be arbitrarily large. In [PR] (cf. also [HS]) the main result is the following:

Theorem 17 ([PR])**.** *If α is a limit ordinal and $1 \leq p < \infty$, then* $\operatorname{rpc}\ell_p(\omega_\alpha) = \omega_\alpha$.

This means of course that $\operatorname{rpc}\ell_p(\omega_\omega) = \omega_\omega$, so $\ell_p(\omega_\omega)$ does not uniformly embed into any $c_0(\Gamma)$. We will generalise this result in Theorem 19 below. The main result in [AKR] is the following:

Theorem 18 ([AKR]). *Let α be 0 or a limit ordinal and $n \in \mathbb{N}_0$. Then* $\operatorname{rpc} \ell_1(\omega_{\alpha+n}) = \omega_\alpha$.

Hence for $\alpha = 0$ and any $n \in \mathbb{N}_0$ we get $\operatorname{rpc} \ell_1(\omega_n) = \omega$, i.e. each of these spaces admit a bi-Lipschitz embedding into some $c_0(\Gamma)$. Let us remark that the proof of the above theorem is based on a combinatorial result [AKLR, Theorem 3], cf. [AKR, Theorem 4]. There seems to be a small mistake in this result: apparently there should be $q = \lfloor \frac{k-1}{l} \rfloor$ instead of $q = \lceil \frac{k-1}{l} \rceil$ in its statement. This means that in fact the cardinal $\varphi(k, l, \alpha)$ is in most cases the successor of the cardinal given in [AKLR, Theorem 3]. Nevertheless, the proof of Theorem 18 in [AKR] can be easily fixed by properly redefining the parameters that relate to [AKLR, Theorem 3]. We will extend this result in Corollary 25 below.

Theorem 19. *Let X be a normed linear space of a non-trivial cotype with* $\operatorname{dens} X \geq \omega_\alpha$, *where α is a limit ordinal. Then* $\operatorname{rpc} X \geq \operatorname{rpc} B_X \geq \omega_\alpha$. *Consequently, if* $\operatorname{dens} X = \omega_\alpha$, *then* $\operatorname{rpc} X = \operatorname{rpc} B_X = \omega_\alpha$.

For the proof we need the following combinatorial lemma from [PR].

Lemma 20 ([PR, Lemma 2]). *Let β be an ordinal and $n \in \mathbb{N}$. Let Γ be a set of cardinality at least $\omega_{\beta+n-1}$, let $\mathcal{C}$ be any set, and let $f : [\Gamma]^n \to \mathcal{C}$ be a colouring such that $f(A) \neq f(B)$ whenever $A, B \in [\Gamma]^n$ are disjoint. Then there exists $\mathcal{H} \subset [\Gamma]^n$ of cardinality ω_β such that* $\operatorname{card} \bigcap \mathcal{H} = n - 1$ *and $f \restriction_{\mathcal{H}}$ is one-to-one.*

Proof of Theorem 19. To prove the inequalities it suffices to show that $\operatorname{rpc} X \geq \omega_\alpha$. Indeed, if $\operatorname{rpc} B_X < \omega_\alpha$, then there is an infinite regular cardinal λ such that $\operatorname{rpc} B_X \leq \lambda < \omega_\alpha$ and hence $\operatorname{rpc} X \leq \lambda$ by Theorem 13, a contradiction.

To prove that $\operatorname{rpc} X \geq \omega_\alpha$ it suffices to show that $\operatorname{ord} \mathcal{V} \geq \omega_\alpha$ for any 1-bounded uniform covering $\mathcal{V}$ of X. So let $\mathcal{V}$ be a 1-bounded uniform covering of X. It suffices to show that for any ordinal $\beta < \alpha$ there is a point in X contained in ω_β sets from $\mathcal{V}$. So let $\beta < \alpha$. Further, let $\delta > 0$ be such that $\mathcal{V}$ is δ-uniform. Let $q \geq 2$ be such that X is of cotype q with cotype-q constant $C > 0$ and set $K = K_{1,q} C$, where $K_{1,q}$ is Kahane's constant for exponents q and 1 (see e.g. [HJ, Theorem 2.76]). Let $n \in \mathbb{N}$ be such that $\frac{1}{K} \sqrt[q]{n} \frac{\delta}{4} > 1$.

Let $\{x_\gamma\}_{\gamma \in \Gamma} \subset S(0, \frac{\delta}{3})$ be a $\frac{\delta}{4}$-separated set with $\Gamma = \operatorname{dens} X$ (see e.g. [HJ, Fact 6.65]). Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2^n}\}$ be some fixed enumeration of the set $\{-1, 1\}^n$. We define a mapping $f : [\Gamma]^n \to \mathcal{V}^{2^n}$ in the following way: Let $A \in [\Gamma]^n$ and let $\gamma_1 < \gamma_2 < \dots < \gamma_n$ be such that $A = \{\gamma_1, \dots, \gamma_n\}$. Put $y_{A,i} = \sum_{j=1}^n \varepsilon_i(j) x_{\gamma_j}$, $i = 1, \dots, 2^n$. Then we set $f(A) = (V_1, \dots, V_{2^n})$, where $V_i \in \mathcal{V}$ are some sets such that $U(y_{A,i}, \delta) \subset V_i$, $i = 1, \dots, 2^n$.

Now assume that $A, B \subset [\Gamma]^n$ are disjoint such that $f(A) = f(B)$. Let $\gamma_1 < \gamma_2 < \dots < \gamma_n$, resp. $\zeta_1 < \zeta_2 < \dots < \zeta_n$ be such that $A = \{\gamma_1, \dots, \gamma_n\}$, resp. $B = \{\zeta_1, \dots, \zeta_n\}$. Then $y_{A,i} - y_{B,i} = \sum_{j=1}^n \varepsilon_i(j)(x_{\gamma_j} - x_{\zeta_j})$, $i = 1, \dots, 2^n$, and so there is $i \in \{1, \dots, 2^n\}$ such that

$$\|y_{A,i} - y_{B,i}\| \geq \frac{1}{K} \left(\sum_{j=1}^n \|x_{\gamma_j} - x_{\zeta_j}\|^q \right)^{\frac{1}{q}} \geq \frac{1}{K} \left(n \frac{\delta^q}{4^q} \right)^{\frac{1}{q}} > 1.$$

Since $\mathcal{V}$ is 1-bounded, it follows that there is no $V \in \mathcal{V}$ such that $y_{A,i} \in V$ and $y_{B,i} \in V$, which contradicts the fact that $f(A) = f(B)$.

Consequently, the assumptions of Lemma 20 are satisfied (notice that $\beta + n - 1 < \alpha$, since α is a limit ordinal). Thus there exists $\mathcal{H} \subset [\Gamma]^n$ of cardinality ω_β such that $\operatorname{card} \bigcap \mathcal{H} = n - 1$ and $f \restriction_{\mathcal{H}}$ is one-to-one. Denote $A = \bigcap \mathcal{H}$ and let $\gamma_1 < \gamma_2 < \dots < \gamma_{n-1}$ be such that $A = \{\gamma_1, \dots, \gamma_{n-1}\}$. Denote $D = \{\gamma \in \Gamma;\ A \cup \{\gamma\} \in \mathcal{H}\}$. Put $\gamma_0 = 0$ and $\gamma_n = \Gamma$. Since $\operatorname{card} D = \omega_\beta$, there is $k \in \{1, \dots, n\}$ such that the ordinal interval (γ_{k-1}, γ_k) contains ω_β elements of D. Denote $E = D \cap (\gamma_{k-1}, \gamma_k)$. Since $f \restriction_{\mathcal{H}}$ is one-to-one, it follows that the set $\mathcal{E} = \{f(A \cup \{\gamma\});\ \gamma \in E\} \subset \mathcal{V}^{2^n}$ has cardinality ω_β. Therefore there is $m \in \{1, \dots, 2^n\}$ such that the projection of $\mathcal{E}$ to the mth coordinate has cardinality ω_β (otherwise $\mathcal{E}$ would be a subset of a finite cartesian product of sets with cardinality smaller that ω_β). By taking one element in the preimage of each point of this projection we conclude that there is $F \subset E$ of cardinality ω_β such that $f(A \cup \{\gamma\})_m \neq f(A \cup \{\zeta\})_m$ whenever $\gamma, \zeta \in F$, $\gamma \neq \zeta$. Pick any $\zeta \in F$ and set $B = A \cup \{\zeta\}$. Since

$$\|y_{B,m} - y_{A \cup \{\gamma\},m}\| = \|\varepsilon_m(k) x_\zeta - \varepsilon_m(k) x_\gamma\| \leq \|x_\zeta\| + \|x_\gamma\| < \delta$$

for any $\gamma \in F$, it follows that $y_{B,m} \in f(A \cup \{\gamma\})_m \subset \mathcal{V}$ for every $\gamma \in F$. All of these sets are however different and this concludes the proof of the inequality $\operatorname{ord} \mathcal{V} \geq \omega_\alpha$.

Finally, if $\operatorname{dens} X = \omega_\alpha$, then we may apply Fact 11.

□

Combining the previous theorem with Theorem 14 we obtain the following corollary:

Corollary 21. *If X is a normed linear space of a non-trivial cotype and* $\operatorname{dens} X \geq \omega_\omega$, *then X does not embed uniformly or coarsely into any $c_0(\Gamma)$.*

On the other hand, in the positive direction we have the following:

Theorem 22. *Let X be a metric space that embeds uniformly into a Hilbert space and let* $\operatorname{dens} X = \omega_{\alpha+n}$, *where α is 0 or a limit ordinal and $n \in \mathbb{N}_0$. Then* $\operatorname{rpc} X \leq \omega_\alpha$. *In particular, if* $\operatorname{dens} X < \omega_\omega$, *then X embeds uniformly into $c_0(\Gamma)$ for some set Γ.*

Proof. Let $\Phi : X \to H$ be a uniform embedding into some Hilbert space H. By considering $\overline{\operatorname{span}}\, \Phi(X)$ we may assume that $\operatorname{dens} H = \operatorname{dens} X$ and $\dim H = \infty$. By [AMM] (see [BL, Corollary 8.11]) the space X embeds uniformly into S_H. Using the classical Mazur mapping ([M], see [BL, Theorem 9.1]) it follows that the space X embeds uniformly into $S_{\ell_1(\Gamma)}$ with card $\Gamma = \operatorname{dens} H = \omega_{\alpha+n}$. Then $\operatorname{rpc} X \leq \omega_\alpha$ by Theorem 18 and Fact 10. If $\operatorname{dens} X < \omega_\omega$, then $\operatorname{rpc} X \leq \omega$, and so X embeds uniformly into $c_0(\Gamma)$ by Theorem 8.

□

We would like to stress that the most important ingredient of the proof above is the fact that we can actually embed X into a sphere of H (or, equivalently, a ball), which allows us to forward it into $\ell_1(\Gamma)$ via the classical Mazur mapping. Another approach is via Theorem 13. For this we will make use of the next two results. The first one was shown in the separable case in [OS]. The proof in the non-separable case is essentially the same, as we explain below.

Theorem 23. *Let X be a Banach space of a non-trivial cotype with a (long) unconditional basis $\{e_\gamma\}_{\gamma\in\Gamma}$. Then B_X is uniformly homeomorphic to $B_{\ell_1(\Gamma)}$.*

Sketch of the proof. As in the first part of the proof of [OS, Theorem 2.1] we deduce that we can assume that X is uniformly convex and uniformly smooth, and that $\{e_\gamma\}_{\gamma\in\Gamma}$ is 1-unconditional. It is easy to see (cf. [OS, Proposition 2.9]) that if $F\colon S_X \to S_Y$ is a uniform homeomorphism between the spheres of two Banach spaces X and Y, then the homogenous extension $\tilde{F}\colon B_X \to B_Y$, $\tilde{F}(x) = \|x\| F(x/\|x\|)$ for $x \neq 0$ and $\tilde{F}(0) = 0$ is a uniform homeomorphism between B_X and B_Y. Thus it is enough to find a uniform homeomorphism between the spheres of X and $\ell_1(\Gamma)$.

For a finite $A \subset \Gamma$ we set $X_A = \operatorname{span}\{e_\gamma;\ \gamma \in A\} \subset X$. From the proof of [OS, Theorem 2.1] we can now deduce the following in the case that card Γ is arbitrary: For each finite $A \subset \Gamma$ there is a uniform homeomorphism $F_A\colon S_{\ell_1(A)} \to S_{X_A}$ with the following properties:

(i) The modulus of uniform continuity of F_A and F_A^{-1} depends only on the modulus of convexity and the modulus of smoothness of X.
(ii) F_A is support preserving and preserves the signs of the coefficients.
(iii) The family $\{F_A\colon A \subset \Gamma \text{ finite}\}$ is consistent, meaning that for $A \subset B$ and $x \in S_{\ell_1(A)}$ it follows that $F_B(x) = F_A(x)$.

From (iii) we deduce that the mapping $F\colon S_{\ell_1(\Gamma)} \cap c_{00}(\Gamma) \to S_X \cap c_{00}(\Gamma)$ defined by $F(x) = F_A(x)$ for $\operatorname{supp} x \subset A$ is well defined. (ii) implies that F is bijective and from (i) we obtain that F and F^{-1} are uniformly continuous. Thus F extends to a uniform homeomorphism between the spheres of $\ell_1(\Gamma)$ and X.

□

Fouad Chaatit proved the following result (cf. also [BL, Theorem 9.7]):

Theorem 24 ([Ch, Theorem 2.1])**.** *Let X be an infinite-dimensional Banach lattice of a non-trivial cotype with a weak unit. Then B_X is uniformly homeomorphic to $B_{\ell_1(\Gamma)}$ for some set Γ.*

As a corollary of the above theorems we obtain an answer to a problem of Christian Avart, Péter Komjáth, and V. Rödl in [AKR]:

Corollary 25. *Let X be an infinite-dimensional subspace of*

(a) $L_p(\mu)$ for some measure μ and $1 \leq p < \infty$, or
(b) a Banach lattice of a non-trivial cotype with a (long) unconditional basis or a weak unit.

Let $\operatorname{dens} X = \omega_{\alpha+n}$*, where α is 0 or a limit ordinal and $n \in \mathbb{N}_0$. Then* $\omega_\alpha = \operatorname{rpc} B_X \leq \operatorname{rpc} X \leq \omega_{\alpha+1}$.

Moreover, $\operatorname{rpc} X = \operatorname{rpc} B_X = \omega_\alpha$ *in the following cases:*

- *(a) holds with $p \leq 2$;*
- *$n = 0$;*
- *ω_α is a regular cardinal (in particular $\alpha = 0$).*

Proof. The case $n = 0$ follows from Theorem 19 when $\alpha > 0$, resp. Fact 11 when $\alpha = 0$. Now assume that $n > 0$. The inequalities $\omega_\alpha \leq \operatorname{rpc} B_X \leq \operatorname{rpc} X$ follow again from Theorem 19. If (a) holds with $p \leq 2$, then $\operatorname{rpc} X \leq \omega_\alpha$ by Theorem 22 and [BL, Example on p. 191]. In the other cases B_X embeds uniformly into $B_{\ell_1(\Gamma)}$ for some set Γ: In the case (a) we use the classical Mazur mapping ([M], see [BL, Theorem 9.1]), in the case (b) we use Theorem 23, resp. 24. By considering $\overline{\operatorname{span}}$ of the embedded image we may assume that card $\Gamma = \operatorname{dens} X = \omega_{\alpha+n}$. Using Theorem 18 and Fact 10 we obtain that $\operatorname{rpc} B_X \leq \omega_\alpha$. Theorem 13 now implies that $\operatorname{rpc} X \leq \omega_\alpha$ in case that ω_α is regular, resp. $\operatorname{rpc} X \leq \omega_{\alpha+1}$ if ω_α is singular.

□

We remark that it is consistent with ZFC that none of the ω_α, α a limit ordinal, is actually regular, see [Je, p. 33].

In combination with Theorem 14 we obtain the following corollary, which contains the solution to a problem of Gilles Godefroy, Gilles Lancien, and Václav Zizler in [GLZ]:

Corollary 26. *Let X be a subspace of*

(a) $L_p(\mu)$ for some measure μ and $1 \leq p < \infty$, or
(b) a Banach lattice of a non-trivial cotype with a (long) unconditional basis or a weak unit.

If $\operatorname{dens} X < \omega_\omega$*, then X admits a bi-Lipschitz embedding into some $c_0(\Gamma)$. If* $\operatorname{dens} X \geq \omega_\omega$*, then X does not admit a coarse or uniform embedding into any $c_0(\Gamma)$.*

The next result follows by the same reasoning as in [HS, Theorem 4.2]. In view of Corollary 26 the density assumption is optimal.

Theorem 27. *If X is a Banach space with a (long) sub-symmetric basis and* $\operatorname{dens} X \geq \omega_\omega$ *which admits a coarse or uniform embedding into $c_0(\Gamma)$, then X is linearly isomorphic to some $c_0(\Lambda)$.*

By combining Corollary 26, [HJ, Theorem 7.63], and [Jo, Theorem 9] we obtain the following corollary (note that any Lipschitz function from a subset of a metric space can be extended to the whole space with the same Lipschitz constant):

Corollary 28. *Let X be a subspace of $L_p(\mu)$ for some measure μ and $1 < p < \infty$, resp. of some super-reflexive Banach lattice with a (long) unconditional basis or a weak unit, with* $\operatorname{dens} X < \omega_\omega$*. Then there is a bi-Lipschitz embedding $\Phi\colon X \to c_0(\Gamma)$ such that the component functions $f_\gamma \circ \Phi$ are C^1-smooth, where f_γ are the canonical coordinate functionals on $c_0(\Gamma)$.*

We do not know whether in the corollary above in the case $p \geq 2$ the component functions can be of higher smoothness. Invoking [HJ, Theorems 7.79 and 7.86] we obtain a corollary on approximation of Lipschitz mappings:

Corollary 29. *Let X be a subspace of $L_p(\mu)$ for some measure μ and $1 < p < \infty$, resp. of some super-reflexive Banach lattice with a (long) unconditional basis or a weak unit, with* $\operatorname{dens} X < \omega_\omega$*, and let Y be a Banach space that is an absolute Lipschitz retract. Then there is a constant $C > 0$ such that for any open $\Omega \subset X$, any L-Lipschitz mapping $f\colon \Omega \to Y$, and any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$ there is a CL-Lipschitz mapping $g \in C^1(\Omega; Y)$ for which $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

The above corollary has applications for Whitney-type extension theorems, see [JZ].

Problem 30. Let X be a super-reflexive space, resp. a WCG space of a non-trivial cotype, with $\operatorname{dens} X < \omega_\omega$. Does X embed uniformly into $c_0(\Gamma)$?

Note that by the result of J. Pelant $C([0, \omega_1])$ does not embed uniformly into $c_0(\Gamma)$, [PHK], so additional assumptions on X must be placed here in order to expect a positive answer to this problem.

It should be noted that the cardinal ω_ω appears rather frequently in dealings with certain properties of the Banach space $c_0(\Gamma)$. For example, in [GKL] it is proved that if X is a WCG space with $\operatorname{dens} X < \omega_\omega$, then the assumption that X is bi-Lipschitz isomorphic to (a subspace of) $c_0(\Gamma)$ implies that it is linearly isomorphic to it. The cardinality restriction is necessary and the result fails for $\operatorname{dens} X = \omega_\omega$, [GLZ]. Similarly, the non-separable Sobczyk theorem holds under the exact same assumptions. More precisely, let $X = c_0(\Gamma)$ be a subspace of a WCG space Y. If $\operatorname{card} \Gamma < \omega_\omega$, then X is complemented. For $\operatorname{card} \Gamma = \omega_\omega$ one can find counterexamples.

We close our paper with another problem related to the implication (ix)⇒(vi) in Theorem 14:

Problem 31. Let X, Y be Banach spaces such that B_X admits a bi-Lipschitz embedding into Y. Does X admit a bi-Lipschitz embedding into Y?

The answer is positive for $Y = c_0(\Gamma)$ (Theorem 14). From the results of Florent Baudier ([B, Theorem 2.2]) it follows that the answer is positive also for spaces Y such that Y is isomorphic to $\ell_p(\mathbb{N}; Y)$ for some $1 \leq p < \infty$, in particular for a Hilbert space, $\ell_p(\Gamma)$, and $L_p(\mu)$ for a σ-finite measure μ. (We would like to thank the referee for pointing F. Baudier's results to us.)

Note also that the uniform version of the above problem has a negative answer: While B_{ℓ_p} is uniformly homeomorphic to B_{ℓ_2} for every $1 < p < \infty$ using the classical Mazur mapping, ℓ_p does not embed uniformly into ℓ_2 for $p > 2$ ([BL, p. 194]).

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