A NOTE ON $C^{1,\alpha}$-SMOOTH APPROXIMATION OF LIPSCHITZ FUNCTIONS

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ABSTRACT. We show that on super-reflexive spaces a Moreau-Yosida type of regularisation by infimal convolution together with a known insertion-type theorem (a variant of Ilmanen's lemma) easily give an approximation of a Lipschitz function by a $C^{1,\alpha}$-smooth Lipschitz function with the same Lipschitz constant. This is a generalisation of the well-known theorem of J.-M. Lasry and P.-L. Lions from Hilbert spaces. It also gives a new self-contained and probably simpler proof of the Lasry-Lions theorem.

In [LL] the authors prove the following theorem: Let H be a Hilbert space, $f: H \to \mathbb{R}$ an L-Lipschitz function, and $\varepsilon > 0$. Then there is an L-Lipschitz function $g \in C^{1,1}(H)$ satisfying $\sup_H |f - g| \le \varepsilon$. (The original formulation in [LL] is for bounded functions, however in this theorem the boundedness is not needed, see e.g. [HJ, Theorem 7.41].) There were attempts to generalise this theorem to super-reflexive spaces (e.g. [C]), but adapting the techniques (especially the use of infimal convolution) from [LL] to more general spaces proved to be very technical and the results produced are weak, namely the approximating function has Hölder derivative only on bounded sets and there is no control whatsoever over the Lipschitz constant. We show that using an insertion theorem we obtain a direct generalisation of the Lasry-Lions theorem; in particular the approximation by $C^{1,1}$-smooth functions holds also e.g. in L_p spaces, $p > 2$. The proof is in fact very simple and arguably even simpler than the proof of the special version from [LL]. (On Hilbert spaces our proof is essentially self-contained: The fact that $\|\cdot\|^2$ is $C^{1,1}$-smooth is almost trivial and there is no need to invoke Lemma 3. The only external result we then use is the fact that a continuous function which is both semiconvex and semiconcave is already smooth – this fact is used also in [LL], albeit without proof or reference.)

1. PRELIMINARIES

A modulus is a non-decreasing function $\omega\colon [0, +\infty) \to [0, +\infty]$ continuous at 0 with $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$. While it is useful to admit infinite values for a modulus when dealing with uniformly continuous functions, for semiconvex functions it is more reasonable to work only with finite moduli, the set of which will be denoted by $\mathcal{M}_{\mathrm{f}}$.

Definition 1. Let X be a normed linear space, $C \subset X$ a convex set, and $\omega \in \mathcal{M}_{\mathrm{f}}$. We say that a function $f: C \to \mathbb{R}$ is semiconvex with modulus ω (or ω-semiconvex for short) if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|$$

for every $x, y \in C$ and $\lambda \in [0, 1]$. We say that $f: C \to \mathbb{R}$ is ω-semiconcave if $-f$ is ω-semiconvex.

For the theory and applications of ω-semiconcave functions in $\mathbb{R}^n$ see [CS].

The next fact follows easily from the definition of ω-semiconvexity ((c) as in the proof of [CS, Proposition 2.1.5]):

Fact 2. *Let X be a normed linear space, $C \subset X$ convex, and $\omega \in \mathcal{M}_{\mathrm{f}}$.*

(a) Let $f: X \to \mathbb{R}$ be ω-semiconvex and $z \in X$. Then the function $x \mapsto f(x+z)$, $x \in X$, is also ω-semiconvex.
(b) Let $f, g\colon C \to \mathbb{R}$ be ω-semiconvex, $a, b \in [0, +\infty)$ and $c \in \mathbb{R}$. Then $af + bg + c$ is semiconvex with modulus $(a+b)\omega$.
(c) Let $\{u_\alpha\}_{\alpha\in\Lambda}$ be a family of ω-semiconvex functions on C. Set $u = \sup_{\alpha\in\Lambda} u_\alpha$ and assume that $u(x) < +\infty$ for each $x \in C$. Then u is also ω-semiconvex.

Let X be a normed linear space and $\omega \in \mathcal{M}$. By $C^{1,\omega}(X)$ we denote the vector space of all Fréchet differentiable $f: X \to \mathbb{R}$ such that the Fréchet derivative Df is uniformly continuous with modulus $C\omega$ for some $C \ge 0$. By $C^{1,\alpha}(X)$, $0 < \alpha \le 1$, we denote the vector space of all Fréchet differentiable functions f on X such that Df is α-Hölder on X, i.e. $C^{1,\alpha}(X) = C^{1,\omega}(X)$ for $\omega(t) = t^\alpha$.

We note that if f is such that Df is uniformly continuous with modulus ω, then f is both ω-semiconvex and ω-semiconcave ([DZ, Lemma 5.2]). The proof of our result is based on the nowadays well-known fact that for continuous functions essentially the converse implication holds (see e.g. [KZ]). In fact, in some rudimentary form this fact is used already in [LL] on Hilbert spaces, although there it is stated without proof or reference (see the discussion in [KZ] for historical details).

We will also make use of the following well-known result:

Lemma 3. *Let X be a Banach space that has an equivalent norm ν with modulus of smoothness of power type $1+\alpha$, $\alpha \in (0, 1]$. Then $\nu^{1+\alpha} \in C^{1,\alpha}(X)$.*

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Proof. Note that passing to an equivalent norm does not change the notion of Fréchet derivative. The mapping $D\nu\colon (X,\nu) \to (X^*,\nu^*)$ is α-Hölder on the unit sphere of (X,ν) (this is implicitly contained in [DGZ, Lemma IV.5.1], for an explicit argument see e.g. [DZ, Lemma 2.6]). [DGZ, proof of Lemma IV.5.9] then yields that $D(\nu^{1+\alpha})\colon (X,\nu) \to (X^*,\nu^*)$ is α-Hölder. Finally, we obtain quite easily that $D(\nu^{1+\alpha})\colon (X,\|\cdot\|) \to (X^*,\|\cdot\|^*)$ is also α-Hölder (see e.g. [JKZ, Lemma 27]). □

Recall that if a Banach space has norm with modulus of smoothness of some power type, then it is super-reflexive, and conversely every super-reflexive space admits an equivalent norm with modulus of smoothness of some power type p, $1 < p \le 2$ (see e.g. [DGZ]).

The approximation in [LL] is based on a Moreau-Yosida regularisation defined by a so-called infimal convolution (this notion goes back to Felix Hausdorff around 1919):

Definition 4. Let X be a vector space and $f, g\colon X \to \mathbb{R}$. We define the infimal convolution of f and g by

$$(f \,\square\, g)(x) = \inf_{y\in X}\bigl(f(y) + g(x-y)\bigr).$$

Note that $f \,\square\, g = g \,\square\, f$.

Fact 5. *Let X be a normed linear space, $f, g\colon X \to \mathbb{R}$, and suppose that f is uniformly continuous with modulus $\omega \in \mathcal{M}$. Then the function $f \,\square\, g$ is uniformly continuous with modulus ω provided that it is everywhere finite.*

Proof. The function $f \,\square\, g = g \,\square\, f$ is an infimum of a system of functions that are all uniformly continuous with modulus ω. □

Fact 6. *Let X be a normed linear space and let $f\colon X \to \mathbb{R}$ be uniformly continuous with modulus $\omega \in \mathcal{M}$. Let $\psi\colon [0,+\infty) \to \mathbb{R}$ be such that $\psi(0) = 0$. Then $g = f \,\square\, (\psi \circ \|\cdot\|)$ satisfies $f \ge g \ge f - \varepsilon$, where $\varepsilon = \sup_{\delta\in[0,+\infty)}\bigl(\omega(\delta) - \psi(\delta)\bigr)$.*

Proof. Fix $x \in X$. Then $g(x) \le f(x) + \psi\bigl(\|x-x\|\bigr) = f(x)$. On the other hand,

$$\begin{aligned} f(x) - g(x) &= f(x) - \inf_{y\in X}\Bigl(f(y) + \psi\bigl(\|x-y\|\bigr)\Bigr) = f(x) + \sup_{y\in X}\Bigl(-f(y) - \psi\bigl(\|x-y\|\bigr)\Bigr) \\ &= \sup_{y\in X}\Bigl(f(x) - f(y) - \psi\bigl(\|x-y\|\bigr)\Bigr) \le \sup_{y\in X}\Bigl(\omega\bigl(\|x-y\|\bigr) - \psi\bigl(\|x-y\|\bigr)\Bigr) \le \sup_{\delta\in[0,+\infty)}\bigl(\omega(\delta) - \psi(\delta)\bigr) = \varepsilon. \end{aligned}$$

□

2. The results

A powerful ingredient of our proof is the following insertion result, which is essentially known. It is a slightly modified [K, Theorem 3.1]:

Proposition 7. *Let X be a normed linear space. Let $f_1, f_2\colon X \to \mathbb{R}$ and $\omega_1, \omega_2 \in \mathcal{M}_{\mathrm{f}}$, $\sigma \in \mathcal{M}$ be such that f_1 is ω_1-semiconvex and uniformly continuous with modulus σ, f_2 is ω_2-semiconcave, and $f_1 \le f_2$. Denote by $\mathcal{S}$ the set of all $s\colon X \to \mathbb{R}$ which are ω_1-semiconvex, uniformly continuous with modulus σ, and satisfy $s \le f_2$. Then the function $f\colon X \to \mathbb{R}$ defined by $f(x) = \sup_{s\in\mathcal{S}} s(x)$ is ω_1-semiconvex, ω_2-semiconcave, uniformly continuous with modulus σ, and satisfies $f_1 \le f \le f_2$.*

The proof is just a copy of the ingenious proof of Václav Kryštof with obvious negligible modifications. However since the proof is very short, we include the full version for reader's convenience. This also makes our note quite self-contained (at least in the case of a Hilbert space).

Proof. By the assumptions $f_1 \in \mathcal{S}$ and so f is well-defined and clearly $f_1 \le f \le f_2$. It is well-known that f is then uniformly continuous with modulus σ. By Fact 2(c) the function f is ω_1-semiconvex. It remains to show that f is ω_2-semiconcave. Fix $u, v \in X$ and $\lambda \in [0,1]$. Set $w = \lambda u + (1-\lambda)v$ and define a function $s\colon X \to \mathbb{R}$ by

$$s(x) = \lambda f(x - w + u) + (1-\lambda) f(x - w + v) - \lambda(1-\lambda)\omega_2(\|u-v\|)\|u-v\| \quad \text{for } x \in X.$$

By Fact 2(a) and (b) the function s is ω_1-semiconvex and it is clearly uniformly continuous with modulus σ. Since f_2 is ω_2-semiconcave, it follows that

$$\begin{aligned} s(x) &\le \lambda f_2(x - w + u) + (1-\lambda) f_2(x - w + v) - \lambda(1-\lambda)\omega_2(\|u-v\|)\|u-v\| \\ &\le f_2\bigl(\lambda(x - w + u) + (1-\lambda)(x - w + v)\bigr) = f_2(x) \end{aligned}$$

for every $x \in X$. Hence $s \in \mathcal{S}$ and consequently $s \le f$. So

$$f(\lambda u + (1-\lambda)v) \ge s(w) = \lambda f(u) + (1-\lambda) f(v) - \lambda(1-\lambda)\omega_2(\|u-v\|)\|u-v\|.$$

□

Corollary 8. *Let X be a normed linear space, $\omega \in \mathcal{M}_{\mathrm{f}}$, $\sigma \in \mathcal{M}$, and let $f_1, f_2\colon X \to \mathbb{R}$ be such that f_1 is ω-semiconvex, f_2 is ω-semiconcave, and $f_1 \le f_2$. If one of the functions f_1, f_2 is uniformly continuous with modulus σ, then there exists $f \in C^{1,\omega}(X)$ that is uniformly continuous with modulus σ and $f_1 \le f \le f_2$. Moreover, Df is uniformly continuous with modulus 4ω.*

Proof. By Proposition 7 there is $f : X \to \mathbb{R}$ that is uniformly continuous with modulus σ, it is both ω-semiconvex and ω-semiconcave, and $f_1 \le f \le f_2$. (In case that f_1 is not uniformly continuous with modulus σ we use Proposition 7 on $-f_2$ and $-f_1$.) Then $f \in C^1(X)$ and Df is uniformly continuous with modulus 4ω by [KZ, Corollary 4.3]. □

Theorem 9. *Let X be a Banach space that has an equivalent norm with modulus of smoothness of power type $1+\alpha$, $\alpha \in (0,1]$. Let $f : X \to \mathbb{R}$ be an L-Lipschitz function. Then for each $\varepsilon > 0$ there is an L-Lipschitz function $g \in C^{1,\alpha}(X)$ such that $\sup_X |f-g| \le \varepsilon$.*

Proof. Let ν be the equivalent norm with modulus of smoothness of power type $1+\alpha$ and let $A > 0$ be such that $\|\cdot\| \le A\nu$. Then f is AL-Lipschitz on the space (X,ν). Let $c > 0$ be such that $ALt - ct^{1+\alpha} \le \varepsilon$ for each $t \ge 0$ (we can take e.g. $c = \frac{(AL)^{1+\alpha}}{\varepsilon^\alpha}$; then $ALt - ct^{1+\alpha} \le ALt \le \varepsilon$ for $t \le \frac{\varepsilon}{AL}$ and $ALt - ct^{1+\alpha} = t(AL - ct^\alpha) \le 0$ for $t \ge \frac{\varepsilon}{AL}$). Set $f_2 = f \,\square\, c\nu^{1+\alpha} + \varepsilon$ and $f_1 = -(-f \,\square\, c\nu^{1+\alpha}) - \varepsilon$. Fact 6 used on the space (X,ν) implies that $f \le f_2 \le f + \varepsilon$. Fact 5 used on the space $(X,\|\cdot\|)$ then implies that f_2 is L-Lipschitz. Since $\nu^{1+\alpha}$ is $C^{1,\alpha}$-smooth (Lemma 3), it follows that $\nu^{1+\alpha}$ is ω-semiconcave with $\omega(t) = kt^\alpha$ for some $k \ge 0$ ([DZ, Lemma 5.2]). Consequently, f_2 is $c\omega$-semiconcave (Fact 2(b), (c)). Similarly, f_1 is $c\omega$-semiconvex and $f - \varepsilon \le f_1 \le f$. It follows that $f - \varepsilon \le f_1 \le f_2 \le f + \varepsilon$. By Corollary 8 there is $g \in C^{1,\alpha}(X)$ that is L-Lipschitz and $f_1 \le g \le f_2$, which implies that $\sup_X |f-g| \le \varepsilon$. □

From [HJ, Theorem 7.86] we obtain the following corollary:

Corollary 10. *Let X be a super-reflexive Banach space and $\Omega \subset X$ an open set. Then for any L-Lipschitz function $f : \Omega \to \mathbb{R}$, any continuous function $\varepsilon : \Omega \to \mathbb{R}^+$, and any $\eta > 1$ there is an ηL-Lipschitz function $g \in C^1(\Omega)$ such that $|f(x) - g(x)| < \varepsilon(x)$ for all $x \in \Omega$.*

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References

[CS] Piermarco Cannarsa and Carlo Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston, Inc., Boston, MA, 2004.

[C] Manuel Cepedello-Boiso, *On regularization in superreflexive Banach spaces by infimal convolution formulas*, Studia Math. **129** (1998), no. 3, 265–284.

[DGZ] Robert Deville, Gilles Godefroy, and Václav Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, Harlow, 1993.

[DZ] Jakub Duda and Luděk Zajíček, *Semiconvex functions: representations as suprema of smooth functions and extensions*, J. Convex Anal. **16** (2009), no. 1, 239–260.

[HJ] Petr Hájek and Michal Johanis, *Smooth analysis in Banach spaces*, De Gruyter Ser. Nonlinear Anal. Appl. 19, Walter de Gruyter, Berlin, 2014.

[JKZ] Michal Johanis, Václav Kryštof, and Luděk Zajíček, *On Whitney-type extension theorems on Banach spaces for $C^{1,\omega}$, $C^{1,+}$, $C^{1,+}_{\mathrm{loc}}$, and $C^{1,+}_{\mathrm{B}}$-smooth functions*, J. Math. Anal. Appl. **532** (2024), Paper No. 127976, 27 pp.

[K] Václav Kryštof, *Generalized versions of Ilmanen lemma: Insertion of $C^{1,\omega}$ or $C^{1,\omega}_{\mathrm{loc}}$ functions*, Comment. Math. Univ. Carolin. **59** (2018), no. 2, 223–231.

[KZ] Václav Kryštof and Luděk Zajíček, *Functions on a convex set which are both ω-semiconvex and ω-semiconcave*, J. Convex Analysis. **29** (2022), 837–856.

[LL] Jean-Michel Lasry and Pierre-Louis Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. **55** (1986), no. 3, 257–266.

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