ON WHITNEY-TYPE EXTENSION THEOREMS ON BANACH SPACES FOR $C^{1,\omega}$, $C^{1,+}$, $C^{1,+}_{\mathrm{loc}}$, AND $C^{1,+}_{\mathrm{B}}$-SMOOTH FUNCTIONS

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ABSTRACT. Our paper is a complement to a recent article by D. Azagra and C. Mudarra (2021). We show how older results on semiconvex functions with modulus ω easily imply extension theorems for $C^{1,\omega}$-smooth functions on super-reflexive Banach spaces which are versions of some theorems of Azagra and Mudarra. We present also some new interesting consequences which are not mentioned in their article, in particular extensions of $C^{1,\omega}$-smooth functions from open quasiconvex sets. They proved also an extension theorem for $C^{1,+}_{\mathrm{B}}$-smooth functions (i.e., functions with uniformly continuous derivative on each bounded set) on Hilbert spaces. Our version of this theorem and new extension results for $C^{1,+}$ and $C^{1,+}_{\mathrm{loc}}$-smooth functions (i.e., functions with uniformly, resp. locally uniformly continuous derivative), all of which are proved on arbitrary super-reflexive Banach spaces, are further main contributions of our paper. Some of our proofs use main ideas of the article by D. Azagra and C. Mudarra, but all are formally completely independent on their article.

1. INTRODUCTION

Our paper is a complement to a recent important article [AM] which concerns Whitney-type extension theorems for $C^{1,\omega}$ and $C^{1,+}_{\mathrm{B}}$-smooth functions (see Definitions 1 and 23) on Hilbert and super-reflexive Banach spaces.

Recall that the famous seminal Whitney's theorem from [Wh1] concerns extensions for C^k-smooth ($k \in \mathbb{N} \cup \{\infty\}$) functions in $\mathbb{R}^n$ and a certain version for $C^{k,\omega}$-smooth functions in $\mathbb{R}^n$ is contained in [Gl] (cf. Remark 3(b)). The first "infinite-dimensional" Whitney-type extension theorem was proved already by J. C. Wells in [We] for $C^{1,1}$-smooth functions (i.e. $C^{1,\omega}$-smooth functions for $\omega(t) = t$) in a Hilbert space. A quite different proof of this surprising theorem was given in [Gr]. Subsequently, a constructive proof of this result was given in [AGM] (where also extension results for *convex* $C^{1,\omega}$-smooth functions on Hilbert and super-reflexive spaces are proved).

Conditions of extendability of a function f in all the articles mentioned above postulate (in the case $k = 1$) the existence of a "candidate to derivative" G such that the "jet" (f, G) has some "Whitney-type" properties (and the theorem asserts that then this jet can be extended to a jet (F, DF)). However, there are also extension theorems for functions on $\mathbb{R}^n$ which do not deal with such G; see [BS2] which deals with $C^{1,\omega}$-smooth functions and papers by C. Fefferman ([F] and papers mentioned therein). No such theorems in infinite-dimensional setting are known.

Our main contributions are extension theorems for $C^{1,+}$ (see Definition 23) and $C^{1,+}_{\mathrm{B}}$-smooth functions (Theorems 57, 6, resp. 59, and 4, resp. 66) which in particular generalise results from [AM] (see [AM, Theorem 6.1 and its proof]) proved in a Hilbert space to an arbitrary super-reflexive Banach space and a new extension theorem for $C^{1,+}_{\mathrm{loc}}$-smooth functions (Theorem 70) which is also proved in an arbitrary super-reflexive Banach space.

Our contribution concerning the $C^{1,\omega}$-smooth extensions from arbitrary sets is not big, since our theorems relatively easily follow from corresponding results of [AM], see Remark 3(d). However, we believe that our proofs of Theorems 2 and 52 which are "qualitative" versions of several theorems from [AM] (which are "quantitative", see Remark 3(d)) are worth publishing since they are very short and transparent as they use older results on ω-semiconvex functions from [DZ] and [Kr1] (see Lemma 39 and Theorem 34 below). Note that these results on ω-semiconvex functions are independently (implicitly) proved in [AM] (in a slightly different setting) using a notion of a "strongly $C\varphi$-paraconvex function", which does not correspond to Rollewicz's notion of strong paraconvexity (cf. Remark 29) but coincides with α-semiconvexity, where $\alpha(t) = C\varphi(t)/t$ for $t > 0$ and $\alpha(0) = 0$. Nevertheless, the main idea of the proofs of Theorems 2 and 52 is taken from [AM], namely we use the crucial observation from [AM] (cf. Lemma 41 and the remark before it) that natural necessary conditions for an extension imply the inequality $h \leq H$ from Lemma 41 (or $m \leq g$ from [AM, p. 9]).

We use the classical Whitney-Glaeser condition (WG_ω) (see Theorem 2) in contrast to conditions $A(f, G) < +\infty$, ($W^{1,\omega}$) and ($mg^{1,\omega}$) used in [AM]. Note that the use of condition (WG_ω) enables us to naturally apply Whitney-type extension results to "true" extension theorems for $C^{1,\omega}$-smooth functions not only on open convex sets, but also on open *quasiconvex* sets (Corollaries 47, 53).

We will use the following basic notions:

Definition 1.

(i) We denote by $\mathcal{M}$ the set of all finite moduli, i.e. the functions $\omega\colon [0, +\infty) \to [0, +\infty)$ which are non-decreasing and continuous at 0 with $\omega(0) = 0$.

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(ii) If U is an open subset of a normed linear space and $\omega \in \mathcal{M}$, we denote by $C^{1,\omega}(U)$ the set of all Fréchet differentiable $f: U \to \mathbb{R}$ such that the Fréchet derivative Df is uniformly continuous with modulus $C\omega$ for some $C \geq 0$, that is

$$\|Df(x) - Df(y)\| \leq C\omega(\|x - y\|) \quad \text{whenever } x, y \in U.$$

Using results of [DZ] and [Kr1] we easily obtain the following result which can be deduced from [AM] (see Remark 3(d)), where, however, no equivalent version is explicitly formulated.

Theorem 2. *Let $\omega \in \mathcal{M}$ be a concave modulus and let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type 2. Let $E \subset X$ and let f be a real function on E. Then f can be extended to a function $F \in C^{1,\omega}(X)$ if and only if the following condition* (WG_ω) *holds:*

There exist a mapping $G: E \to X^*$ and $M > 0$ such that

$$\|G(y) - G(x)\| \leq M\omega(\|y - x\|),$$
$$\bigl|f(y) - f(x) - \langle G(x), y - x\rangle\bigr| \leq M\omega(\|y - x\|)\|y - x\|$$

for each $x, y \in E$.

Moreover, if (WG_ω) *is satisfied, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.*

The assumption on X cannot be relaxed, see Remark 54(a). By Remark 37, Theorem 2 (and its Corollary 47) can be applied e.g. if X is isomorphic to some $L_p(\mu)$ with $p \geq 2$.

Remark 3.

(a) Condition (WG_ω) naturally corresponds to Whitney's condition in his celebrated theorem from [Wh1] on extensions to C^k-smooth functions in $X = \mathbb{R}^n$ (for $k = 1$) and appears explicitly in Glaeser's thesis [Gl].

(b) It was stated several times that Theorem 2 was proved for $X = \mathbb{R}^n$ by Glaeser in [Gl] (see e.g. [BS1, p. 516], [AM, p. 2]). However, [Gl] (which works with $C^{k,\omega}$-smooth functions) contains only a corresponding result on extension of a function f defined on a closed subset F of a compact interval $K \subset \mathbb{R}^n$ ("pavé compact") to a function belonging to $C^{1,\omega}(K)$ (see [Gl, Proposition VII, p. 33, $m = 1$]; cf. corresponding [M, Complement 3.6, p. 9]).

(c) Whitney-Glaeser condition (WG_ω) is equivalent to condition [AM, (1.2)] (for $X = \mathbb{R}^n$), which we obtain from (WG_ω) by changing "$\leq M\omega(\|y - x\|)\|y - x\|$" to "$\leq M\varphi(\|y - x\|)$", where $\varphi(t) = \int_0^t \omega(s)\,\mathrm{d}s$ for $t \geq 0$ (cf. Fact 13 below).

(d) If X is a Hilbert space, then the statement of Theorem 2 follows from [AM, Theorem 1.6]. Moreover, in full generality it follows from [AM, Theorem 1.9] using our Lemma 41, Lemma 27, and the easy observation from Remark 40 below. Further, results in [AM] assert not only that $C^{1,\omega}$-smooth extension F exists, but they are quantitative in the sense that they assert that DF is uniformly continuous with modulus $K\omega$, where $K > 0$ (which does not depend on f) is explicitly given. Examining the proofs of Theorem 34 and Lemma 39 we could obtain similar quantitative results for Theorems 2 and 52 too, but less precise than those in [AM] which seem to be close to the (unknown) optimal ones.

We moreover prove Theorem 52, an extension theorem for $C^{1,\alpha}$-smooth functions which works e.g. in $L_p(\mu)$ spaces with $p \geq 1 + \alpha$. Theorem 52 can be also deduced from [AM]. However, our (simple but important) observation that the assumption on X in Theorem 2 and Theorem 52 cannot be relaxed (see Remark 54) is new.

We are also interested in the extension of $C^{1,\omega}$-smooth functions from open sets, which is not considered in [AM]. As a consequence of Theorems 2 and 52 we obtain new interesting extension theorems from so-called open quasiconvex (and in particular convex) sets. These results do not use condition (WG_ω) and so they are true extension theorems for $C^{1,\omega}$-smooth functions. (The set is called quasiconvex if its inner metric is bi-Lipschitz equivalent to the metric of the enclosing space; see Definition 18. Quasiconvex sets are sometimes called "sets with Whitney arc property", since they were used in Whitney's article [Wh2] concerning extensions of C^k-smooth functions in $\mathbb{R}^n$.) Namely, in Corollary 47 we prove that if X is as in Theorem 2 and $\omega \in \mathcal{M}$, then each function from $C^{1,\omega}(U)$, $U \subset X$ quasiconvex, can be extended to a function from $C^{1,\omega}(X)$; see also analogous Corollary 53.

Another new result is Theorem 66 on extensions for functions from $C^{1,+}_{\mathrm{B}}(X)$ (i.e. functions with uniformly continuous derivative on every bounded set), which is in fact a generalisation of [AM, Theorem 6.1] from Hilbert spaces to arbitrary super-reflexive spaces. A simplified formulation is as follows:

Theorem 4. *Let X be a super-reflexive Banach space. Let $E \subset X$ and let f be a real function on E. Then f can be extended to a real function $F \in C^{1,+}_{\mathrm{B}}(X)$ if and only if the following condition* ($\widetilde{\mathrm{W}}$) *holds:*

There exists a mapping $G: E \to X^*$ such that for each bounded $B \subset E$ the function f is bounded on B, the mapping G is bounded and uniformly continuous on B, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\bigl|f(y) - f(x) - \langle G(x), y - x\rangle\bigr| \leq \varepsilon\|y - x\|$ whenever $x, y \in B$ and $\|y - x\| < \delta$.

Moreover, if ($\widetilde{\mathrm{W}}$) *is satisfied, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.*

Remark 5. Using the compactness of bounded closed sets in $\mathbb{R}^n$ it is easy to see that $C^{1,+}_{\mathrm{B}}(\mathbb{R}^n) = C^1(\mathbb{R}^n)$ and if E is closed, then condition ($\widetilde{\mathrm{W}}$) is equivalent to the condition of the classical Whitney's C^k extension theorem for $k = 1$ (see [Wh1, p. 34, (3.2)]), which is also easily seen to be equivalent to the condition (E) from [JS]. However, Theorem 4 (resp. C^1-extension theorem [JS, Theorem A.2]) implies only a part of Whitney's C^1-extension theorem, since the latter asserts that the extension is C^∞-smooth (even analytic) on the complement of the (closed) set E. In fact it is an open problem whether stronger versions of Theorem 2, resp. 4 (and so more precise analogues of Whitney's theorem) hold, namely whether the extension can be C^∞-smooth (or at least C^2-smooth) on the complement of a (closed) set E.

We also prove two results (Theorems 57 and 59) on extensions from arbitrary sets for $C^{1,+}$-smooth functions. A simplified formulation of Theorem 59 is as follows:

Theorem 6. *Let X be a super-reflexive Banach space, $E \subset X$, and let f be a real Lipschitz function on E. Then f can be extended to a Lipschitz function $F \in C^{1,+}(X)$ if and only if there exists a bounded uniformly continuous $G\colon E \to X^*$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\big|f(y) - f(x) - \langle G(x), y - x\rangle\big| \le \varepsilon\|y - x\|$$

whenever $x, y \in E$ and $\|y - x\| < \delta$.

By Remark 69 the assumption of super-reflexivity of X in Theorems 4 and 6 cannot be relaxed. As a corollary of Theorems 57 and 59 we obtain extension results for $C^{1,+}$-smooth functions on open quasiconvex sets (Corollaries 61, 62, and 67). Another interesting theorem that has no analogue in [AM] is Theorem 70, which in particular characterises those functions on a closed subset of a super-reflexive space that can be extended to functions on the whole space having locally uniformly continuous derivative.

The structure of the present article is following: In (unfortunately rather long) Section 2 we fix the notation and recall the needed known facts on moduli, ω-semiconvex functions, and super-reflexive spaces. The core of the article is contained in a short Section 3, where we prove two basic lemmata and one proposition. Using results of Section 3 we rather easily prove Theorem 2 and other results concerning the $C^{1,\omega}$ case in Section 4. In Sections 5 and 6 we prove new results concerning $C^{1,+}$, $C^{1,+}_{\mathrm{B}}$, $C^{1,+}_{\mathrm{loc}}$ cases, in particular Theorem 66 (which contains Theorem 4).

We remark that all the positive extension results are proved in super-reflexive spaces. Remarks 54, 69 and 71 show that for all these results this assumption on X is necessary.

Some natural questions remain open, see Remarks 5, 51, 58, 64, and 65.

2. Preliminaries

2.1. **Basic notation, properties of moduli, and quasiconvex sets.** All the normed linear spaces considered are real. Let $(X, \|\cdot\|)$ be a normed linear space. By $U(x,r) \subset X$, resp. $B(x,r) \subset X$, resp. $S(x,r) \subset X$ we denote the open ball, resp. closed ball, resp. sphere centred at $x \in X$ with radius $r > 0$. By B_X, resp. S_X we denote the closed unit ball, resp. unit sphere in X. All the canonical norms on different spaces are denoted by $\|\cdot\|$, even on the dual space, if it cannot lead to confusion. For $x \in X$ and $g \in X^*$ we will denote the evaluation of g at x also by $\langle g, x\rangle = g(x)$. An L-Lipschitz mapping is a mapping with a (not necessarily minimal) Lipschitz constant L. By $Df(x)$ we will denote the Fréchet derivative of f at x, its evaluation in the direction h will be denoted by $Df(x)[h]$. We remind that the set of all finite moduli $\mathcal{M}$ and the $C^{1,\omega}$-smooth functions are defined in Definition 1. Sometimes we will work with several norms on a space. If it is necessary to specify objects with respect to a particular norm we will use the notation $U_{\|\cdot\|}(0,r)$, $C^{1,\omega}_{\|\cdot\|}$, etc.

For a mapping $f\colon X \to Y$, where X is a set and Y is a vector space, we denote $\operatorname{supp_o} f = f^{-1}(Y \setminus \{0\})$. If X is a topological space, then we denote $\operatorname{supp} f = \overline{\operatorname{supp_o} f}$.

For the (standard) definition and properties of the length (variation) of a continuous curve $\gamma\colon [0,1] \to X$ in a metric space X see e.g. [Ch].

Recall that a system $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of functions on a set X is called a partition of unity if

- $\psi_\alpha\colon X \to [0,1]$ for all $\alpha \in \Lambda$,
- $\sum_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for each $x \in X$.

We say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is subordinated to a covering $\mathcal{U}$ of X if $\{\operatorname{supp_o}\psi_\alpha\}_{\alpha\in\Lambda}$ refines $\mathcal{U}$, i.e. for each $\alpha \in \Lambda$ there is $U \in \mathcal{U}$ such that $\operatorname{supp_o}\psi_\alpha \subset U$. Further, in case that X is a topological space we say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite if the system $\{\operatorname{supp_o}\psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite, i.e. if for each point $x \in X$ there is a neighbourhood U of x such that the set $\{\alpha \in \Lambda;\ \operatorname{supp_o}\psi_\alpha \cap U \neq \emptyset\}$ is finite.

Definition 7. Let (X,ρ), (Y,σ) be metric spaces and $f\colon X \to Y$. Then:

(i) The *minimal modulus of continuity of f* is the function $\omega_f\colon [0,+\infty) \to [0,+\infty]$ defined in the usual way:
$$\omega_f(\delta) = \sup\big\{\sigma(f(x), f(y));\ x, y \in X, \rho(x,y) \le \delta\big\}.$$
(ii) We say that f is *uniformly continuous with modulus $\omega \in \mathcal{M}$* if $\sigma(f(x), f(y)) \le \omega(\rho(x,y))$ for every $x, y \in X$, i.e. $\omega_f \le \omega$.

(iii) We say that f is *α-Hölder* ($\alpha > 0$) if f is uniformly continuous with modulus $\omega(t) = Ct^\alpha$ for some $C > 0$.

Clearly, ω_f is non-decreasing, and the mapping f is uniformly continuous if and only if $\lim_{t\to 0+}\omega_f(t) = 0$.

Recall that $\omega\colon [0,+\infty) \to [0,+\infty]$ is *sub-additive* if $\omega(s+t) \le \omega(s) + \omega(t)$ for any $s, t \in [0,+\infty)$. For ω sub-additive it immediately follows by induction that

$$\omega(Nt) \le N\omega(t) \quad \text{for each } t \ge 0,\ N \in \mathbb{N}. \tag{1}$$

Consequently, the following fact holds:

Fact 8. *If* $\omega\colon [0,+\infty) \to [0,+\infty]$ *is sub-additive and finite on a neighbourhood of* 0*, then it is finite everywhere.*

The following fact is easy and well-known (it follows from [AP, Remark 1, p. 407] and Fact 8):

Fact 9. *If A is a convex subset of a normed linear space, Y is a metric space, and $f\colon A \to Y$ is uniformly continuous, then ω_f is sub-additive and $\omega_f \in \mathcal{M}$.*

The next two standard facts are also very easy to see.

Fact 10. *If $\omega \in \mathcal{M}$ is concave, then it is sub-additive and $\omega(Kt) \leq K\omega(t)$ for any $t \geq 0$, $K \geq 1$.*

Fact 11. *If C is a convex subset of a normed linear space, Y is a metric space, and $f_1, \dots, f_n\colon C \to Y$ are uniformly continuous, then there is $\omega \in \mathcal{M}$ such that all $f_1, \dots, f_n$ are uniformly continuous with modulus ω.*

Proof. It suffices to set $\omega = \omega_{f_1} + \dots + \omega_{f_n}$ and notice that $\omega \in \mathcal{M}$ by Fact 9. □

We will also use the following well-known facts.

Lemma 12. *Let $\omega \in \mathcal{M}$.*

(i) If ω satisfies $\limsup_{t\to+\infty} \frac{\omega(t)}{t} < +\infty$, then there exists a concave $\widetilde{\omega} \in \mathcal{M}$ such that $\omega \leq \widetilde{\omega}$.
(ii) If ω is bounded, then there exists a bounded concave $\widetilde{\omega} \in \mathcal{M}$ such that $\omega \leq \widetilde{\omega}$.
(iii) If ω is sub-additive, then there exists a concave $\widetilde{\omega} \in \mathcal{M}$ such that $\omega \leq \widetilde{\omega} \leq 2\omega$.

The proof of (i) is quite easy, see [AP, proof of Theorem 1, p. 407], where the theorem states sub-additivity of $\widetilde{\omega}$ but in fact the proof shows even concavity. Alternatively one can use the statement of [AP, Theorem 1, p. 406] together with (iii). Statement (ii) follows easily from (i). Fact (iii) is due to Stechkin, but was not published by him. For a proof, see [E, p. 78] or [Ko, p. 670].

Fact 13. *Let $\omega \in \mathcal{M}$. If we set $\varphi(t) = \int_0^t \omega(s)\,\mathrm{d}s$ for $t \geq 0$, then $\varphi \in \mathcal{M}$ and φ is convex. If moreover ω is concave, then $t\omega(t) \leq 2\varphi(t)$ for every $t \geq 0$.*

Proof. The convexity of φ follows from the fact that ω is non-decreasing, see e.g. [RV, p. 11, p. 13]. Now suppose that ω is concave. Then $\frac{\omega(t)}{t} \leq \frac{\omega(s)}{s}$ for each $0 < s \leq t$, and so

$$t\omega(t) = 2\int_0^t \frac{s}{t}\omega(t)\,\mathrm{d}s \leq 2\int_0^t \omega(s)\,\mathrm{d}s = 2\varphi(t).$$

□

Definition 14. A metric space (X,ρ) is called *almost convex* if for any $x, y \in X$ and $s, t > 0$ such that $\rho(x,y) < s + t$ there is $z \in X$ such that $\rho(x,z) \leq s$ and $\rho(z,y) \leq t$.

Note that this notion goes back to Aronszajn's (unpublished) thesis; see [AP, p. 417], where the name "almost 3-hyperconvex metric space" is used. The following (almost obvious) fact is mentioned in [AP, p. 438, footnote 20].

Fact 15. *Let (X,ρ) be a metric space where every two points can be connected by a rectifiable curve and let ρ_{i} be the corresponding inner (intrinsic) metric. Then (X,ρ_{i}) is an almost convex metric space.*

The article [BV] contains (without a proof) the following interesting and probably new observation which we will use below in the proof of Lemma 21. (Note that the assumption of uniform continuity of f was forgotten in [BV]. We have been informed by Taras Banakh that this assumption cannot be dropped, cf. Remark 17.)

Proposition 16. *Let (X,ρ), (Y,σ) be metric spaces and let $f\colon X \to Y$ be uniformly continuous. If X is almost convex, then ω_f is sub-additive.*

Proof. Let $a, b \in [0,+\infty)$ and $\varepsilon > 0$. Let $\delta > 0$ be such that $\sigma(f(u), f(v)) \leq \varepsilon$ whenever $\rho(u,v) \leq \delta$. Pick any $x, y \in X$ such that $\rho(x,y) \leq a+b < a+b+\frac{\delta}{2}$. Using the almost convexity we can find $u \in X$ such that $\rho(x,u) \leq a$ and $\rho(u,y) \leq b+\frac{\delta}{2} < b+\delta$. Using the almost convexity again we can find $v \in X$ such that $\rho(u,v) \leq \delta$ and $\rho(v,y) \leq b$. Hence

$$\sigma(f(x), f(y)) \leq \sigma(f(x), f(u)) + \sigma(f(u), f(v)) + \sigma(f(v), f(y)) \leq \omega_f(a) + \varepsilon + \omega_f(b).$$

It follows that $\omega_f(a+b) \leq \omega_f(a) + \omega_f(b) + \varepsilon$, and since this holds for any $\varepsilon > 0$, the sub-additivity follows. □

Remark 17. If (X,ρ) is totally convex (which is a stronger property then almost convexity), then the assumption of uniform continuity of f is unnecessary, see [AP, Remark 1].

However, the assumption of uniform continuity of f cannot be dropped not only in Proposition 16 but also in the case when (X,ρ) is an inner metric space. Indeed let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ be equipped with the euclidean metric ρ which clearly coincides with ρ_{i}. Set $f(-1,0) = 0$; $f(x,y) = 1$ if $x \leq 0$, $(x,y) \neq (-1,0)$, $(x,y) \neq (0,0)$; $f(x,y) = 2$ if $x > 0$, $(x,y) \neq (1,0)$; and $f(1,0) = 3$. Then clearly $\omega_f(2) = 3 > 2 = \omega_f(1) + \omega_f(1)$.

The following notion of a quasiconvex space (or domain in $\mathbb{R}^n$) is nowadays a standard tool in Geometric Analysis.

Definition 18. We say that a metric space (X, ρ) is *c-quasiconvex* for $c \geq 1$ if for each $x, y \in X$ there exists a continuous rectifiable curve $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{len}\gamma \leq c\rho(x,y)$, where $\operatorname{len}\gamma$ is the length (variation) of the curve γ. We say that X is *quasiconvex* if it is c-quasiconvex for some $c \geq 1$.

Note that convex subsets of normed linear spaces are 1-quasiconvex.

Remark 19. It is well-known and easy to prove that each bi-Lipschitz image of a quasiconvex metric space is quasiconvex.

The following example, which is essentially well-known (cf. [V1, Lemma 5.9]), will be used later in Remark 69.

Example 20. Let X be a normed linear space such that $\dim X \geq 2$ and let $0 \leq r < R \leq +\infty$. Then the spherical shell $D = \{x \in X;\ r < \|x\| < R\}$ is 2-quasiconvex. Indeed, let $x, y \in D$. We may assume that $\|y\| \geq \|x\|$. Let $t = \max\{s \in [0,1];\ \|x + s(y-x)\| \leq \|x\|\}$. By [Sch, Theorem 4J] there is a continuous rectifiable curve γ joining x and $x + t(y-x)$ in $S(0, \|x\|) \subset D$ such that $\operatorname{len}\gamma \leq 2t\|y-x\|$. By concatenating γ with the segment $[x + t(y-x), y] \subset D$ we obtain a continuous rectifiable curve joining x and y in D whose length is at most $2t\|x-y\| + (1-t)\|x-y\| \leq 2\|x-y\|$.

There is a plenty of non-convex quasiconvex sets, e.g. bi-Lipschitz images of convex sets and spherical shells are quasiconvex (by Remark 19).

Lemma 21. *Let A be a c-quasiconvex subset of a normed linear space and let $f\colon A \to Y$ be a uniformly continuous mapping into a metric space (Y, σ). Then there exists a concave $\omega \in \mathcal{M}$ such that $\frac{1}{2}\omega \leq \omega_f \leq c\omega$.*

Proof. Let ρ_{i} be the inner metric on A induced by the norm. Note that $\|x-y\| \leq \rho_{\mathrm{i}}(x,y) \leq c\|x-y\|$ for each $x, y \in A$. Then the mapping $f\colon (A, \rho_{\mathrm{i}}) \to Y$ is uniformly continuous; denote by ω_f^{i} its minimal modulus of continuity. Proposition 16 together with Fact 15, and Fact 8 imply that ω_f^{i} is sub-additive and finite. Now Lemma 12(iii) implies that there is a concave $\omega \in \mathcal{M}$ such that $\omega_f^{\mathrm{i}} \leq \omega \leq 2\omega_f^{\mathrm{i}}$. Since $\|x-y\| \leq \rho_{\mathrm{i}}(x,y)$, it follows that $\frac{1}{2}\omega \leq \omega_f^{\mathrm{i}} \leq \omega_f$. Further, using Fact 10 we obtain

$$\sigma(f(x), f(y)) \leq \omega_f^{\mathrm{i}}(\rho_{\mathrm{i}}(x,y)) \leq \omega(\rho_{\mathrm{i}}(x,y)) \leq \omega(c\|y-x\|) \leq c\omega(\|y-x\|)$$

for each $x, y \in A$ and so $\omega_f \leq c\omega$. □

The following application of quasiconvexity is well-known:

Proposition 22. *Let X, Y be normed linear spaces, $U \subset X$ an open c-quasiconvex set, and let $f\colon U \to Y$ be a Fréchet differentiable mapping. If $K = \sup_{x \in U}\|Df(x)\| < +\infty$, then f is cK-Lipschitz.*

Proof. The convexity of neighbourhoods of points in U implies that f is locally K-Lipschitz. The statement now follows from [V2, Lemma 5.5]. □

2.2. **Subclasses of C^1-smooth functions and Whitney-type conditions.** Besides the class of $C^{1,\omega}$-smooth functions we will consider also the following subclasses of C^1-smooth functions:

Definition 23. Let X be a normed linear space and $U \subset X$ open. We denote by $C^{1,\alpha}(U)$, $0 < \alpha \leq 1$, the set of all Fréchet differentiable functions f on U such that Df is α-Hölder on U, i.e. $C^{1,\alpha}(U) = C^{1,\omega}(U)$ for $\omega(t) = t^\alpha$. We denote by $C^{1,+}(U)$ the set of all Fréchet differentiable functions f on U such that Df is uniformly continuous on U, and by $C^{1,+}_{\mathrm{B}}(U)$ the set of all Fréchet differentiable functions f on U such that Df is uniformly continuous on each bounded subset of U. Further, by $C^{1,+}_{\mathrm{loc}}(U)$ we denote the set of all functions on U which are locally $C^{1,+}$-smooth on U.

For the reader's convenience we now recall two Whitney-type conditions (WG_ω) and ($\widetilde{\mathrm{W}}$) used in Theorem 2 and 4 in Section 1 and introduce a third one, (W_G).

Definition 24. Let X be a normed linear space, $E \subset X$, $\omega \in \mathcal{M}$, and $f\colon E \to \mathbb{R}$. We say that f satisfies condition

(W_G) if $G\colon E \to X^*$ is uniformly continuous and for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\left|f(y) - f(x) - \langle G(x), y-x\rangle\right| \leq \varepsilon\|y-x\|$$
whenever $x, y \in E$ and $\|y-x\| < \delta$;

(WG_ω) if there exist a mapping $G\colon E \to X^*$ and $M > 0$ such that
$$\|G(y) - G(x)\| \leq M\omega(\|y-x\|),$$
$$\left|f(y) - f(x) - \langle G(x), y-x\rangle\right| \leq M\omega(\|y-x\|)\|y-x\|$$
for each $x, y \in E$;

($\widetilde{\mathrm{W}}$) if there exists a mapping $G\colon E \to X^*$ such that for each bounded $B \subset E$ the function f is bounded on B, the mapping G is bounded and uniformly continuous on B, and for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\left|f(y) - f(x) - \langle G(x), y-x\rangle\right| \leq \varepsilon\|y-x\|$$
whenever $x, y \in B$ and $\|y-x\| < \delta$.

Using only the continuity of ω at 0 it is easy to see that

$$\text{if } f \text{ satisfies } (\mathrm{WG}_\omega) \text{ with some } G, \text{ then it also satisfies } (\mathrm{W}_G). \tag{2}$$

For some other connections between the above conditions see Lemmata 55, 56, and Theorem 66. The motivation for the second inequality in (WG_ω) is the following fact which is an immediate consequence of the Taylor formula (see e.g. [HJ, Corollary 1.108]).

Lemma 25. *Let U be an open convex subset of a normed linear space, $f \in C^1(U)$, and suppose that the derivative Df is uniformly continuous with modulus $\omega \in \mathcal{M}$. Then*

$$\bigl|f(y) - f(x) - Df(x)[y-x]\bigr| \le \omega(\|y-x\|)\|y-x\|$$

for each $x, y \in U$.

Indeed, Lemma 25 clearly implies the following fact:

Fact 26. *Let U be an open convex subset of a normed linear space, $\omega \in \mathcal{M}$, and $f \in C^{1,\omega}(U)$. Then f satisfies condition* (WG_ω) *(with $G = Df$).*

This fact will be generalised to quasiconvex sets in Lemma 46.

Sometimes we will work with renormed normed linear spaces, so we need to know whether certain properties or notions (e.g. $C^{1,\omega}$-smoothness) are invariant with respect to renormings:

Lemma 27. *Let $(X, \|\cdot\|)$ be a normed linear space, $V \subset X$, $\omega \in \mathcal{M}$, and let $|||\cdot|||$ be an equivalent norm on X.*

(i) Assume that ω is sub-additive or V is convex. Let $H : (V, \|\cdot\|) \to (X^*, \|\cdot\|^*)$ be uniformly continuous with modulus ω. Then there is $K > 0$ such that $H : (V, |||\cdot|||) \to (X^*, |||\cdot|||^*)$ is uniformly continuous with modulus $K\omega$.

(ii) Assume that V is open, and ω is sub-additive or V is convex. Then $C^{1,\omega}_{\|\cdot\|}(V) = C^{1,\omega}_{|||\cdot|||}(V)$.

(iii) Assume that V is open. Then $C^{1,+}_{\|\cdot\|}(V) = C^{1,+}_{|||\cdot|||}(V)$.

(iv) Assume that ω is sub-additive and let $f : V \to \mathbb{R}$. Then f satisfies condition (WG_ω) *on V in the space $(X, \|\cdot\|)$ if and only if f satisfies* (WG_ω) *on V in the space $(X, |||\cdot|||)$ (with the same mapping G).*

Proof. Let $A \in \mathbb{N}$ be such that $\|\cdot\| \le A|||\cdot|||$.

(i) Pick any $x, y \in V$. If ω is sub-additive, then by (1)

$$|||H(x) - H(y)|||^* \le A\|H(x) - H(y)\|^* \le A\omega(\|x-y\|) \le A\omega(A|||x-y|||) \le A^2\omega(|||x-y|||).$$

If V is convex let us denote by $\omega_H^{\|\cdot\|}$ the minimal modulus of continuity of H under the metrics given by $\|\cdot\|$ and $\|\cdot\|^*$. Then $\omega_H^{\|\cdot\|}$ is sub-additive (Fact 9), and therefore by the first part of the proof

$$|||H(x) - H(y)|||^* \le A^2\omega_H^{\|\cdot\|}(|||x-y|||) \le A^2\omega(|||x-y|||).$$

(ii) Note that by the symmetry it suffices to show just one inclusion. Let $f \in C^{1,\omega}_{\|\cdot\|}(V)$ and let $L > 0$ be such that $Df : (V, \|\cdot\|) \to (X^*, \|\cdot\|^*)$ is uniformly continuous with modulus $L\omega$. Then $Df : (V, |||\cdot|||) \to (X^*, |||\cdot|||^*)$ is uniformly continuous with modulus $KL\omega$ by (i) and so $f \in C^{1,\omega}_{|||\cdot|||}(U)$.

(iii) Note that by the symmetry it suffices to show just one inclusion. Let $f \in C^{1,+}_{\|\cdot\|}(V)$. Choose any $\varepsilon > 0$. There is $\delta > 0$ such that $\|Df(x) - Df(y)\|^* < \frac{\varepsilon}{A}$ whenever $x, y \in V$, $\|x-y\| < \delta$. Now if $x, y \in V$ are such that $|||x-y||| < \frac{\delta}{A}$, then $\|x-y\| \le A|||x-y||| < \delta$ and so $|||Df(x) - Df(y)|||^* \le A\|Df(x) - Df(y)\|^* < \varepsilon$.

(iv) By the symmetry it suffices to show just one implication. So suppose that f satisfies condition (WG_ω) on V in the space $(X, \|\cdot\|)$ with some $G : V \to X^*$ and $M > 0$. By (i) there is $K > 0$ such that $|||G(x) - G(y)|||^* \le KM\omega(|||x-y|||)$ for each $x, y \in V$. Further,

$$\bigl|f(y) - f(x) - \langle G(x), y-x\rangle\bigr| \le M\omega(\|y-x\|)\|y-x\| \le AM\omega(A|||y-x|||)|||y-x||| \le A^2M\omega(|||y-x|||)|||y-x|||$$

for each $x, y \in V$. Thus f satisfies (WG_ω) on V in the space $(X, |||\cdot|||)$ with G and the constant $\widetilde{M} := \max\{A^2M, KM\}$. □

2.3. Semiconvex functions with modulus ω and super-reflexive Banach spaces.

Definition 28. Let X be a normed linear space, $U \subset X$ an open convex set, and $\omega \in \mathcal{M}$. We say that a function $f : U \to \mathbb{R}$ is semiconvex with modulus ω (or ω-semiconvex for short) if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) + \lambda(1-\lambda)\omega(\|x-y\|)\|x-y\|$$

for every $x, y \in U$ and $\lambda \in [0,1]$. We say that $f : U \to \mathbb{R}$ is ω-semiconcave if $-f$ is ω-semiconvex.

Remark 29. For the theory and applications of ω-semiconcave functions in $\mathbb{R}^n$ see [CS]. For a comparison of ω-semiconvexity with the closely related Rollewicz's notion of strong paraconvexity (which works with $\min\{\lambda, 1-\lambda\}$ instead of $\lambda(1-\lambda)$), see e.g. [DZ, Remark 2.11] or [JouThZa, p. 218].

On infinite-dimensional spaces ω-semiconvex functions need not be continuous (consider a discontinuous linear functional). Similarly, the "strong $C\varphi$-paraconvexity" used in [AM] does not imply continuity. Consequently [AM, Lemma 2.8] is not correct as stated; apparently the assumption that h is locally bounded is omitted in its assumptions (without this assumption the proof using [JouThZa, Proposition 6.1] does not work). Also the applications of [AM, Proposition 2.6] in several places need additional explanation: It is used on functions which are not a priori continuous, e.g. the function F in the proof of [AM, Lemma 3.5]. However, from definitions of these functions it easily follows that they are locally bounded and so continuous by [JouThZa, Proposition 6.1]. An alternative explanation (noted by the referee of our paper) uses the observation that the proof of [AM, Proposition 2.6] works also if the assumption of continuity of u is weakened to local boundedness of u only. After this change no use of [JouThZa, Proposition 6.1] (or Proposition 30 below) is necessary.

Note that, as for the continuity, ω-semiconvex functions behave similarly as convex functions. The following well-known proposition is an analogue of the classical results for convex functions. (Later we will use the assumption (ii), although the application of (i) would be also possible.)

Proposition 30. *Let X be a normed linear space, $U \subset X$ an open convex set, $\omega \in \mathcal{M}$, and $f : U \to \mathbb{R}$ an ω-semiconvex function. Suppose that one of the following conditions hold:*

(i) f is locally bounded.

(ii) X is a Banach space and f is lower semi-continuous.

Then f is locally Lipschitz (and hence continuous).

Proof. (i) Observe that f is strongly $\alpha(\cdot)$-paraconvex for $\alpha(t) = t\omega(t)$ with constant $C = 1$ (it follows from the inequality $t(1-t) \le \min\{t, 1-t\}$, $t \in [0, 1]$, cf. [DZ, Remark 2.11] or [JouThZa, p. 218]), so the statement follows by [R, Proposition 5]. Alternatively we can use [JouThZa, Proposition 6.1(b)], since each ω-semiconvex function on an open ball is clearly strongly 1-paraconvex.

(ii) Let $x_0 \in U$. Choose $\delta > 0$ such that $B(x_0, \delta) \subset U$ and set $g = f$ on $B(x_0, \delta)$ and $g = +\infty$ on $X \setminus B(x_0, \delta)$. It is easy to check that g is a proper lower semi-continuous function that is approximately convex on U in the sense of [NLT]. So the statement follows by [NLT, Proposition 3.2].

□

The next fact follows easily from the definition of ω-semiconvexity (as in the proof of [CS, Proposition 2.1.5]):

Fact 31. *Let X be a normed linear space, $U \subset X$ open and convex, and $\omega \in \mathcal{M}$. Let $\{u_\alpha\}_{\alpha \in \Lambda}$ be a family of ω-semiconvex functions on U. Set $u = \sup_{\alpha \in \Lambda} u_\alpha$ and assume that $u(x) < +\infty$ for each $x \in U$. Then u is also ω-semiconvex.*

For the following well-known fact see e.g. [KZ, (5), p. 838] or [DZ, Lemma 5.2].

Lemma 32. *Let X be a normed linear space, $U \subset X$ open convex set, and let $f : U \to \mathbb{R}$ be Fréchet differentiable such that Df is uniformly continuous on U with modulus $\omega \in \mathcal{M}$. Then f is ω-semiconvex on U.*

Remark 33. In the preceding lemma we can also clearly assert that f is ω-semiconcave. On the other hand if f is continuous and both ω-semiconvex and ω-semiconcave, then for some U (e.g. $U = X$) it follows that $f \in C^{1,\omega}(U)$. For more information see [KZ] and [AM, Proposition 2.6].

The first basic ingredient of our paper is the following insertion theorem which is a version of the well-known Ilmanen's lemma.

Theorem 34 ([Kr1, Corollary 3.2])**.** *Let X be a normed linear space, $\omega \in \mathcal{M}$, and let $f_1 \le f_2$ be continuous functions on X such that f_1 is ω-semiconvex and f_2 is ω-semiconcave. Then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \le f \le f_2$.*

Remark 35.

(a) Theorem 34 is independently (implicitly and in a different language) proved in [AM, Proof of Theorem 3.2].

(b) Using [Kr1, Theorem 3.1] and [KZ, Corollary 4.3] we can assert that Df is uniformly continuous with modulus 4ω in Theorem 34.

We now recall some facts we need about super-reflexive Banach spaces. For the original definition of super-reflexive spaces and several of their characterisations see e.g. [DGZ] or [BL]. For example a Banach space is super-reflexive if and only if it has an equivalent norm which is uniformly smooth.

Recall that if X is a Banach space, then the *modulus of smoothness* of the norm of X (or of the space X) is defined as

$$\rho(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1;\ x, y \in X, \|x\| = \|y\| = 1\right\}, \quad \tau \ge 0.$$

Notice that $\rho(\tau) \le \tau$ for any $\tau \ge 0$. We say that the modulus of smoothness is of power type $p > 1$ if there exists $K > 0$ such that $\rho(\tau) \le K\tau^p$ for each $\tau \ge 0$; see e.g. [DGZ]. Note that if ρ is of power type $p > 1$, then it is also of power type q for any $1 < q < p$ (recall that $\rho(\tau) \le \tau \le \tau^q$ for $\tau \ge 1$). For the following Pisier's result see e.g. [DGZ] or [BL, p. 412].

Theorem 36. *A Banach space is super-reflexive if and only if it has an equivalent norm with modulus of smoothness of power type p for some $1 < p \le 2$.*

Remark 37. It is well-known (see [DGZ, Corollary V.1.2]) that if μ is any measure, then $L_p(\mu)$ has modulus of smoothness of power type p (resp. 2) if $1 < p \le 2$ (resp. $p \ge 2$). In particular, every Hilbert space has modulus of smoothness of power type 2.

We will need also the following well-known fact (see e.g. [DZ, Lemma 2.6] for an argument).

Lemma 38. *If the norm $\|\cdot\|$ of a Banach space X has modulus of smoothness of power type $1+\alpha$ for some $0<\alpha\le 1$, then the Fréchet derivative $D\|\cdot\|$ exists and is α-Hölder on S_X.*

The following lemma, which is the second basic ingredient of our paper, is a direct consequence of [DZ, Lemma 5.3] (together with Lemma 38).

Lemma 39. *Let $\omega \in \mathcal{M}$ be a concave modulus and let X be a super-reflexive Banach space whose norm has modulus of smoothness of power type $1+\alpha$ for some $0<\alpha\le 1$. Denote $\varphi(t)=\int_0^t \omega(s)\,\mathrm{d}s$ for $t\ge 0$ and $\nu(x)=\varphi(\|x\|)$ for $x\in X$. Then the following assertions hold:*

(i) If ω is bounded, then $\nu \in C^{1,\sigma}(X)$ for some bounded $\sigma\in\mathcal{M}$.
(ii) If $\alpha=1$, then $\nu\in C^{1,\omega}(X)$.
(iii) If $\omega(t)=ct^{\beta}$ for some $0<\beta\le\alpha$ and $c>0$, then $\nu\in C^{1,\omega}(X)=C^{1,\beta}(X)$.

Indeed, it is sufficient to use Lemma 38 and then [DZ, Lemma 5.3] with $\beta:=\alpha$, $\alpha:=\beta$, $\varphi:=\omega$, and $\psi:=\varphi$. The items (i), (ii), (iii) follow from items (ii), (iv), (v) of [DZ, Lemma 5.3], respectively.

Remark 40. Assertion (ii) and a slight generalisation of (iii) are essentially proved in [AM]. Namely the proof of [AM, Lemma 3.6] shows that $\nu\in C^{1,\omega}(X)$ whenever the function $p(t):=t^{\alpha}/\omega(t)$, $t>0$, is non-decreasing, which clearly implies (iii) and also (ii), since for $\alpha=1$ the function p is non-decreasing, because the concavity of ω implies that $\omega(t)/t=1/p(t)$ is non-increasing. However, this latter fact is not mentioned explicitly in [AM].

3. Basic lemmata and a proposition

The third basic ingredient of our paper is the following easy but non-trivial fact that is (without the explicit constant 6) implicitly almost contained in [AM] without a proof or a reference (cf. first two lines of p. 3 and the note just after Definition 1.4).

Lemma 41. *Let X be a normed linear space, $E\subset X$, let $\omega\in\mathcal{M}$ be concave, and suppose that $f\colon E\to\mathbb{R}$ satisfies condition* (WG_ω) *on E with a mapping $G\colon E\to X^*$ and $M>0$. Let $\varphi(t)=\int_0^t\omega(s)\,\mathrm{d}s$ for $t\ge0$. For each $z\in E$ set*

$$\begin{aligned} h_z(x) &= f(z)+\langle G(z),x-z\rangle-6M\varphi(\|x-z\|),\\ H_z(x) &= f(z)+\langle G(z),x-z\rangle+6M\varphi(\|x-z\|),\quad x\in X, \end{aligned}$$

and define

$$h(x)=\sup_{z\in E}h_z(x),\quad H(x)=\inf_{z\in E}H_z(x),\quad x\in X.$$

Then $h(x)\le H(x)$ for each $x\in X$.

Proof. Let $x\in X$ and $z_1,z_2\in E$. It is sufficient to show that $h_{z_1}(x)\le H_{z_2}(x)$, which clearly follows from

$$\big|f(z_1)-f(z_2)+\langle G(z_1),x-z_1\rangle-\langle G(z_2),x-z_2\rangle\big|\le 6M\varphi(\|x-z_1\|)+6M\varphi(\|x-z_2\|).$$

To prove this inequality we may assume that $\|x-z_1\|\le\|x-z_2\|$. Using the concavity of ω via Fact 10 and Fact 13 (twice) we obtain

$$\begin{aligned} \omega(\|z_1-z_2\|)\|z_1-z_2\| &\le 4\tfrac12\omega\big(\tfrac12\|z_1-z_2\|\big)\|z_1-z_2\|\le 8\varphi\big(\tfrac12\|z_1-z_2\|\big)\\ &\le 8\varphi\big(\tfrac12\|x-z_1\|+\tfrac12\|x-z_2\|\big)\le 4\varphi(\|x-z_1\|)+4\varphi(\|x-z_2\|) \end{aligned}$$

and using sub-additivity of ω (Fact 10) and Fact 13 we obtain

$$\begin{aligned} \omega(\|z_1-z_2\|)\|x-z_1\| &\le \big(\omega(\|x-z_1\|)+\omega(\|x-z_2\|)\big)\|x-z_1\|\\ &\le \omega(\|x-z_1\|)\|x-z_1\|+\omega(\|x-z_2\|)\|x-z_2\|\le 2\varphi(\|x-z_1\|)+2\varphi(\|x-z_2\|). \end{aligned}$$

Thus

$$\begin{aligned} \big|f(z_1)-f(z_2)+\langle G(z_1),x-z_1\rangle-\langle G(z_2),x-z_2\rangle\big| &= \big|f(z_1)-f(z_2)-\langle G(z_2),z_1-z_2\rangle+\langle G(z_1)-G(z_2),x-z_1\rangle\big|\\ &\le M\omega(\|z_1-z_2\|)\|z_1-z_2\|+M\omega(\|z_1-z_2\|)\|x-z_1\|\\ &\le 6M\varphi(\|x-z_1\|)+6M\varphi(\|x-z_2\|). \end{aligned}$$

□

All our extension results are based on the proposition below (which is a consequence of Lemma 41 and Theorem 34) and Lemma 39. This proposition with $\sigma=\omega$ is essentially mentioned in [AM] (see the first sentence after [AM, Theorem 1.10]) and is essentially implicitly used in [AM]. The case $\sigma\ne\omega$ together with Lemma 39(i) will be used substantially in Section 5.

Proposition 42. *Let $\omega\in\mathcal{M}$ be concave, $\varphi(t)=\int_0^t\omega(s)\,\mathrm{d}s$ for $t\ge0$, and $\sigma\in\mathcal{M}$. Let X be a Banach space such that the function $\nu(x)=\varphi(\|x\|)$, $x\in X$, is $C^{1,\sigma}$-smooth. Let f be a real function on $E\subset X$ which satisfies condition* (WG_ω) *on E with some mapping $G\colon E\to X^*$. Then f can be extended to a function $F\in C^{1,\sigma}(X)$ such that $DF(x)=G(x)$ for each $x\in E$.*

Proof. Let $M > 0$ be as in condition (WG$_\omega$). For each $z \in E$ set

$$h_z(x) = f(z) + \langle G(z), x - z\rangle - 6M\varphi(\|x - z\|),$$
$$H_z(x) = f(z) + \langle G(z), x - z\rangle + 6M\varphi(\|x - z\|), \quad x \in X,$$

and define

$$h(x) = \sup_{z\in E} h_z(x), \quad H(x) = \inf_{z\in E} H_z(x), \quad x \in X.$$

Then $h \le H$ by Lemma 41. Let $x \in E$. Since $h_x(x) = H_x(x) = f(x)$, it follows that $H(x) \le f(x) \le h(x)$ and consequently $h(x) = f(x) = H(x)$.

By the assumptions there exists $K > 0$ such that $D\nu$ is uniformly continuous with modulus $K\sigma$, which easily implies that for each $z \in E$ the derivative Dh_z is uniformly continuous with modulus $6MK\sigma$. So each function h_z, $z \in E$, is continuous and semiconvex with modulus $6MK\sigma$ by Lemma 32 and so also (clearly finite) function h is semiconvex with modulus $6MK\sigma$ by Fact 31. Further, h is clearly lower semi-continuous, and so it is continuous by Proposition 30(ii). Quite analogously (working with functions $-H_z$ and $-H = \sup_{z\in E} -H_z$) we obtain that H is semiconcave with modulus $6MK\sigma$ and continuous. So Theorem 34 implies that there exists $F \in C^{1,6MK\sigma}(X) = C^{1,\sigma}(X)$ such that $h \le F \le H$. Since $h(x) = H(x) = f(x)$ for $x \in E$, it follows that F is an extension of f.

Further, fix $z \in E$. Then $h_z(x) \le h(x) \le F(x) \le H(x) \le H_z(x)$ for each $x \in X$. Since $D\nu(0)$ exists by the assumption and ν has a minimum at 0, it is clear that $D\nu(0) = 0$. Therefore $Dh_z(z) = G(z) = DH_z(z)$. Since also $h_z \le F \le H_z$ and $h_z(z) = F(z) = H_z(z)$, it is easy to see that $DF(z) = G(z)$.

□

Remark 43. The smoothness of ν and the introduction of functions h_z and h (with $M = 1/6$) was essential already in [DZ] in connection with extensions of ω-semiconvex functions; see [DZ, Lemma 5.4 and Proposition 5.12].

Remark 44. In fact the completeness of X in the previous proposition is not necessary. The continuity of h and H can be alternatively easily deduced from Proposition 30(i).

Remark 45. The space X from the assumptions of Proposition 42 is necessarily super-reflexive, provided that $\omega \ne 0$. Indeed, let $a > 0$. Then $\varphi(a) > 0$. Let $\psi \in C^{1,1}(\mathbb{R})$ be such that $\psi(t) = 0$ for $t \ge \varphi(a)$ and $\psi(0) > 0$. Then $\psi \circ \nu$ is a $C^{1,+}$-smooth bump (use [HJ, Proposition 1.128](ii) on a sufficiently large ball). Consequently X is super-reflexive by [DGZ, Theorem V.3.2].

The following fact is a basic tool for our extension results from open quasiconvex sets.

Lemma 46. *Let U be an open c-quasiconvex subset of a normed linear space X, $\omega \in \mathcal{M}$, and $f \in C^{1,\omega}(U)$. Then f satisfies condition* (WG$_\omega$) *with $G = Df$ (and $M = 4Kc^3$, where $K > 0$ is such that $\omega_{Df} \le K\omega$).*

Proof. Consider an arbitrary $K > 0$ such that $\omega_{Df} \le K\omega$. By Lemma 21 there is a concave $\widetilde{\omega} \in \mathcal{M}$ such that $\frac{1}{2}\widetilde{\omega} \le \omega_{Df} \le c\widetilde{\omega}$. Note that $\widetilde{\omega} \le 2\omega_{Df} \le 2K\omega$. Consider arbitrary different points $x, y \in U$. Then we can choose a continuous $\gamma\colon [0,1] \to U$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $0 < \|x - y\| \le L := \operatorname{len}\gamma \le c\|x - y\|$. Set $\varepsilon = \operatorname{dist}(\gamma([0,1]), X \setminus U)$. Then $\varepsilon > 0$ by the compactness of $\gamma([0,1])$. By the uniform continuity of γ we choose $\delta > 0$ such that $\|\gamma(s) - \gamma(t)\| < \varepsilon$ whenever $0 \le s \le t \le 1$ and $t - s < \delta$. Further, choose points $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1$ such that $t_i - t_{i-1} < \delta$ for $i = 1, \dots, n$ and denote $x_i = \gamma(t_i)$. Then clearly $x_0 = x$, $x_n = y$, and

$$\|x_j - x_0\| \le \sum_{i=1}^{n} \|x_i - x_{i-1}\| \le L, \quad j = 1, \dots, n. \tag{3}$$

The choice of ε, δ, and $t_0, \dots, t_n$ implies that $x_i \in U(x_{i-1}, \varepsilon) \subset U$ for $i = 1, \dots, n$, and so Lemma 25 together with $\omega_{Df} \le c\widetilde{\omega}$ and (3) imply that

$$\big|f(x_i) - f(x_{i-1}) - Df(x_{i-1})[x_i - x_{i-1}]\big| \le c\widetilde{\omega}(\|x_i - x_{i-1}\|)\|x_i - x_{i-1}\| \le c\widetilde{\omega}(L)\|x_i - x_{i-1}\|$$

for $i = 1, \dots, n$. Further, $\omega_{Df} \le c\widetilde{\omega}$ with (3) gives $\|Df(x_j) - Df(x_0)\| \le c\widetilde{\omega}(\|x_j - x_0\|) \le c\widetilde{\omega}(L)$ for $j = 1, \dots, n$. Using these inequalities, (3), the concavity of $\widetilde{\omega}$, and $c \ge 1$ we obtain

$$\begin{aligned}
\big|f(y) - f(x) - Df(x)[y - x]\big| &= \left|\sum_{i=1}^{n}\big(f(x_i) - f(x_{i-1})\big) - \sum_{i=1}^{n} Df(x_0)[x_i - x_{i-1}]\right| \\
&\le \left|\sum_{i=1}^{n}\big(f(x_i) - f(x_{i-1}) - Df(x_{i-1})[x_i - x_{i-1}]\big)\right| + \left|\sum_{i=1}^{n}\big(Df(x_{i-1}) - Df(x_0)\big)[x_i - x_{i-1}]\right| \\
&\le \sum_{i=1}^{n} c\widetilde{\omega}(L)\|x_i - x_{i-1}\| + \sum_{i=1}^{n} c\widetilde{\omega}(L)\|x_i - x_{i-1}\| \le 2c\widetilde{\omega}(L)L \\
&\le 2c^2\widetilde{\omega}(c\|y - x\|)\|y - x\| \le 2c^3\widetilde{\omega}(\|y - x\|)\|y - x\| \le 4Kc^3\omega(\|y - x\|)\|y - x\|,
\end{aligned}$$

and so f satisfies (WG$_\omega$) with $G = Df$ and $M = 4Kc^3$, since $\omega_{Df} \le K\omega \le M\omega$.

□

4. THE $C^{1,\omega}$ CASE

The main result of this section is Theorem 2. For the convenience of the reader we will repeat the statement.

Theorem 2. *Let $\omega \in \mathcal{M}$ be a concave modulus and let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type 2. Let $E \subset X$ and let f be a real function on E. Then f can be extended to a function $F \in C^{1,\omega}(X)$ if and only if f satisfies condition* (WG_ω) *on E. Moreover, if* (WG_ω) *is satisfied with a $G\colon E \to X^*$, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.*

The assumption on X cannot be relaxed, see Remark 54(a).

Proof. $\Rightarrow$ This implication holds in fact in an arbitrary normed linear space X. Indeed, Fact 26 implies that F satisfies (WG_ω) on X and so f satisfies (WG_ω) on E.

$\Leftarrow$ Choose an equivalent norm $|||\cdot|||$ on X with modulus of smoothness of power type 2. Then f satisfies condition (WG_ω) with respect to the norm $|||\cdot|||$ with some G (Lemma 27(iv)). The function $\nu = \varphi \circ |||\cdot|||$, where $\varphi(t) = \int_0^t \omega(s)\,\mathrm{d}s$, is $C^{1,\omega}$-smooth on $(X, |||\cdot|||)$ by Lemma 39(ii). So we can apply Proposition 42 with $\sigma = \omega$ and extend f to a function F on X which is $C^{1,\omega}$-smooth on $(X, |||\cdot|||)$, and hence also on $(X, \|\cdot\|)$ by Lemma 27(ii). Moreover, Proposition 42 gives that $DF(x) = G(x)$ for each $x \in E$. □

As a corollary we obtain the following true extension theorem.

Corollary 47. *Let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type 2. Let $U \subset X$ be an open quasiconvex set, $\omega \in \mathcal{M}$, and $f \in C^{1,\omega}(U)$. Then f can be extended to a function $F \in C^{1,\omega}(X)$.*

The assumption on X cannot be relaxed, see Remark 69.

Proof. There is $K > 0$ such that $\omega_{Df} \le K\omega$. By Lemma 21 there are a concave $\widetilde{\omega} \in \mathcal{M}$ and $c \ge 1$ such that $\frac{1}{2}\widetilde{\omega} \le \omega_{Df} \le c\widetilde{\omega}$. Therefore $f \in C^{1,\widetilde{\omega}}(U)$ and Lemma 46 then implies that f satisfies condition $(\mathrm{WG}_{\widetilde{\omega}})$ on U. Thus f can be extended to a function $F \in C^{1,\widetilde{\omega}}(X)$ by Theorem 2. Since $\widetilde{\omega} \le 2\omega_{Df} \le 2K\omega$, it follows that $F \in C^{1,\omega}(X)$. □

Corollary 47 for $X = \mathbb{R}^n$ and $\omega(t) = t$, $t \ge 0$, "almost follows" from [BB, Theorem 2.64]; for the details see the following remark.

Remark 48. (a) [BB, Theorem 2.64] gives that if $\omega \in \mathcal{M}$ is concave, $U \subset \mathbb{R}^n$ is a (C,ω)-convex domain, $k \ge 1$, and $f \in \dot{C}^{k,\omega}(U)$, then f has an extension $F \in \dot{C}^{k,\omega}(\mathbb{R}^n)$. Note that if $f \in C^{1,\omega}(U)$ and both f and Df are bounded, then clearly $f \in \dot{C}^{1,\omega}(U)$ (and the opposite implication holds for $U = \mathbb{R}^n$). Further, if $\omega(t) = t$ for $t \ge 0$, then (C,ω)-convexity coincides with C-quasiconvexity. Using these facts we obtain that the assertion of Corollary 47 follows from [BB, Theorem 2.64] if $X = \mathbb{R}^n$, $\omega(t) = t$ for $t \ge 0$, and both f and Df are bounded on U.

(b) However, this connection between Corollary 47 and [BB, Theorem 2.64] fails for some more general moduli. For example, if $0 < \alpha < 1$ is fixed and $\omega_\alpha(t) = t^\alpha$ for $t \ge 0$, then the set of all open quasiconvex sets in $\mathbb{R}^n$ $(n > 1)$ is strictly larger than the set of all (C,ω_α)-convex domains with any $C > 0$. Indeed, suppose that $U \subset \mathbb{R}^n$ is a (C,ω_α)-convex domain, i.e. for each $x, y \in U$ there exists a "polygonal line $\gamma\colon [0,1] \to U$ joining x and y with the segments $[\gamma(t_i), \gamma(t_{i+1})]$, where $0 = t_0 < t_1 < \dots < t_k = 1$" such that

$$\ell_{\omega_\alpha}(\gamma) := \sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|^\alpha \le C\,\|x-y\|^\alpha.$$

Then

$$\operatorname{len}\gamma = \sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| \le \left(\sum_{i=0}^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|^\alpha\right)^{\frac{1}{\alpha}} \le C^{\frac{1}{\alpha}}\,\|x-y\|.$$

(For the first inequality recall that ω_α is sub-additive.) Therefore U is $C^{\frac{1}{\alpha}}$-quasiconvex. Note that this fact that we just proved contradicts the incorrect claim from [BB] that the set $G_{\frac{1}{\alpha}}$ from [BB, Figure 2.1] (cf. Remark 63), which is clearly not quasiconvex, is (C,ω_α)-convex. For another argument that this claim is incorrect see Remark 63.

Further, in the case $\alpha = \frac{1}{2}$ (other cases are similar) we can define an open quasiconvex set $U \subset \mathbb{R}^2$ which is not a (C,ω_α)-convex domain for any C by

$$U = \left\{(x,y) \in \mathbb{R}^2;\ 0 < x < 1, x^2 \sin\frac{1}{x} < y < x^4 + x^2 \sin\frac{1}{x}\right\}.$$

It is not too difficult to prove that U has the aforementioned properties.

Remark 49. It is well-known that Corollary 47 (resp. Corollary 53 below) does not hold for general simply connected domains U (i.e. we cannot relax the assumption of quasiconvexity of U to being simply connected) even when $X = \mathbb{R}^2$ and $\omega \in \mathcal{M}$ is arbitrary concave non-zero (resp. $0 < \alpha \le 1$). Since we were not able to find a reference to a corresponding construction, we supply the following easy one: Let $U = (0,3) \times (0,3) \setminus \big([1,2] \times [1,2] \cup (0,1) \times \{2\}\big) \subset \mathbb{R}^2$ and define $f : U \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{for } (x,y) \in (0,3) \times (0,1) \cup (0,1) \times [1,2), \\ 1 & \text{for } (x,y) \in (0,3) \times (2,3), \\ \sin^2\big(\frac{\pi}{2}(y-1)\big) & \text{for } (x,y) \in (2,3) \times [1,2]. \end{cases}$$

Then $f \in C^{1,1}(U) \subset C^{1,\omega}(U)$, but it cannot be extended even to a continuous function on $\overline{U}$.

Now we prove a proposition which works with arbitrary (possibly *non-concave*) moduli $\omega \in \mathcal{M}$ and may be regarded as a generalisation of Theorem 2.

Proposition 50. *Let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type* 2*. Let $E \subset X$, let f be a real function on E, and let $\omega \in \mathcal{M}$. Then f can be extended to a function $F \in C^{1,\omega}(X)$ if and only if there exists a concave $\psi \in \mathcal{M}$ such that $\psi \le \omega$ and f satisfies* (WG_ψ) *on E. Moreover, if f satisfies* (WG_ψ) *with some G, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.*

Proof. $\Rightarrow$ Let $K > 0$ be such that $\omega_{DF} \le K\omega$. The modulus ω_{DF} is sub-additive (Fact 9) and so there is a concave $\widetilde{\omega} \in \mathcal{M}$ such that $\omega_{DF} \le \widetilde{\omega} \le 2\omega_{DF} \le 2K\omega$ (Lemma 12(iii)). Thus we can define concave modulus $\psi = \frac{1}{2K}\widetilde{\omega}$ for which $\psi \le \omega$ and $F \in C^{1,\psi}(X)$. Fact 26 implies that F satisfies (WG_ψ) on X and so f satisfies (WG_ψ) on E.

$\Leftarrow$ By Theorem 2 the function f can be extended to $F \in C^{1,\psi}(X) \subset C^{1,\omega}(X)$ and the "moreover" part holds. □

In connection with the previous proposition, in the following remark we consider the validity of Theorem 2 for non-concave moduli. The main observation is that the assumption of concavity cannot be completely dropped.

Remark 51. (a) Assume that $\omega \in \mathcal{M}$ is not concave and Theorem 2 holds for this ω. Then there is a non-zero concave $\psi \in \mathcal{M}$ such that $\psi \le \omega$ (which is clearly equivalent to the fact that $\liminf_{t\to 0+} \frac{\omega(t)}{t} > 0$). Indeed, consider $e \in X$, $e \neq 0$. Set $f(0) = 0$, $f(e) = 0$, $f(2e) = 1$. Then f clearly satisfies (WG_ω) (e.g. with $G = 0$) and so can be extended to $F_1 \in C^{1,\omega}(X)$. By Proposition 50 there exists a concave $\psi \in \mathcal{M}$ such that $\psi \le \omega$ and f satisfies (WG_ψ). Thus by Theorem 2 the function f can be extended to $F_2 \in C^{1,\psi}(X)$. It follows that ψ is non-zero. Otherwise DF_2 would be constant and in turn F_2 would be affine, which is a contradiction with the definition of f.

So we have shown that Theorem 2 does not hold for arbitrary $\omega \in \mathcal{M}$.

(b) Of course there are some non-concave moduli ω for which Theorem 2 holds. In fact it is easy to see that it is sufficient to assume that there exist a concave $\psi \in \mathcal{M}$ and $K > 0$ such that $\psi \le \omega \le K\psi$, which holds e.g. if ω is sub-additive (Lemma 12(iii)). However, we do not know whether this condition is necessary.

Other interesting consequences of Proposition 42 are the following extension results for functions with Hölder derivatives.

Theorem 52. *Let $0 < \alpha \le 1$ and let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type $1+\alpha$. Let $E \subset X$ and let f be a real function on E. Then f can be extended to a function $F \in C^{1,\alpha}(X)$ if and only if condition* (WG_ω) *holds for $\omega(t) = t^\alpha$.*

Moreover, if (WG_ω) *is satisfied with a $G\colon E \to X^*$, then F can be found such that $DF(x) = G(x)$ for $x \in E$.*

The assumption on X cannot be relaxed, see Remark 54(a).

Proof. $\Rightarrow$ follows from Fact 26 in an arbitrary normed linear space as in the proof of Theorem 2.

$\Leftarrow$ Choose an equivalent norm $|||\cdot|||$ on X with modulus of smoothness of power type $1+\alpha$. Then f satisfies condition (WG_ω) with respect to the norm $|||\cdot|||$ with some G (Lemma 27(iv)). The function $\nu = \varphi \circ |||\cdot|||$, where $\varphi(t) = \int_0^t \omega(s)\,\mathrm{d}s$, is $C^{1,\omega}$-smooth on $(X, |||\cdot|||)$ by Lemma 39(iii). So we can apply Proposition 42 with $\sigma = \omega$ and extend f to a function F on X which is $C^{1,\omega}$-smooth on $(X, |||\cdot|||)$, and hence also on $(X, \|\cdot\|)$ by Lemma 27(ii). Moreover, Proposition 42 gives that $DF(x) = G(x)$ for each $x \in E$. □

By Remark 37, Theorem 52 (and its Corollary 53) can be applied e.g. if X is isomorphic to some $L_p(\mu)$ with $p \ge 1+\alpha$.

Corollary 53. *Let $0 < \alpha \le 1$ and let X be a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type $1+\alpha$. Let $U \subset X$ be an open quasiconvex set and $f \in C^{1,\alpha}(U)$. Then f can be extended to a function $F \in C^{1,\alpha}(X)$.*

The assumption on X cannot be relaxed, see Remark 69.

Proof. Set $\omega(t) = t^\alpha$ for $t \ge 0$. Lemma 46 implies that f satisfies condition (WG_ω) on U and so f can be extended to a function $F \in C^{1,\alpha}(X)$ by Theorem 52. □

Remark 54. Let X be a normed linear space and let $\omega \in \mathcal{M}$ be such that $\omega(t) > 0$ for $t > 0$. Assume that we can extend every function from an arbitrary closed subset E of X satisfying condition (WG_ω) on E to a function in $C^{1,\omega}(X)$. Then we obtain the following smooth separation property: For every closed $A \subset X$ and $\delta > 0$ there is a function $\varphi \in C^{1,\omega}(X)$ such that $\varphi = 1$ on A and $\varphi(x) = 0$ whenever $\operatorname{dist}(x, A) \geq \delta$. Indeed, set $f = 1$, $G = 0$ on A and $f = 0$, $G = 0$ on $B = \{x \in X;\ \operatorname{dist}(x, A) \geq \delta\}$. Then f satisfies condition (WG_ω) with G and $M = \frac{1}{\delta\omega(\delta)}$ on the closed set $A \cup B$ and so it can be extended to a desired function $\varphi \in C^{1,\omega}(X)$. In particular, we can produce a $C^{1,\omega}$-smooth bump function on X and consequently X is super-reflexive by [DGZ, Theorem V.3.2]. (A similar observation is already known, cf. [AM, p. 9].)

In the Hölder case, i.e. $\omega(t) = t^\alpha$ for some $0 < \alpha \leq 1$, we obtain two new interesting applications.

(a) The assumption on X in Theorems 2 and 52 (i.e. the existence of an equivalent norm with modulus of smoothness of power type 2, resp. $1 + \alpha$) cannot be relaxed (for Theorem 2 consider $\omega(t) = t$). This follows again from [DGZ, Theorem V.3.2].

(b) The above separation property gives the existence of a locally finite $C^{1,\alpha}$-smooth partition of unity on an arbitrary open $U \subset X$ subordinated to an arbitrary open covering of U. Indeed, we use [HJ, Lemma 7.49, (ii)⇒(vi)] with $S = \{\varphi \in C^{1,\alpha}(U);$ φ is bounded and Lipschitz$\}$. Then S is a partition ring on U, which can be seen by repeating the beginning of the proof of [HJ, Theorem 7.56] with obvious changes. To verify condition [HJ, Lemma 7.49(ii)], given $V \subset W \subset U$ bounded open sets satisfying $\operatorname{dist}(V, U \setminus W) > 0$ we set $A = \overline{V}$, $\delta = \operatorname{dist}(V, U \setminus W)$ and use the separation property above. This gives us a corresponding $\psi \in C^{1,\alpha}(X)$ and since it has a bounded support, both ψ and its derivative are bounded and so ψ is Lipschitz. Then $\varphi = \psi\restriction_U \in S$.

Therefore as a corollary of Theorem 52 we obtain the existence of locally finite $C^{1,\alpha}$-smooth partitions of unity on (open subsets of) super-reflexive spaces that have an equivalent norm with modulus of smoothness of power type $1 + \alpha$. The proof we just described is much easier than the original proof in [JohToZi] which uses some deep and heavy machinery of non-separable Banach space theory. So we believe that the above observation is interesting in itself.

5. The $C^{1,+}$ case

First we observe two more connections (deeper than (2)) between the Whitney-Glaeser type conditions from Definition 24.

Lemma 55. *Let X be a normed linear space, let f be a function on $E \subset X$, and let $G\colon E \to X^*$. Then the following statements are equivalent:*

(i) f satisfies condition (W_G) *and for each $d > 0$ there exists $K > 0$ such that*
$$\|G(y) - G(x)\| \leq K\|y - x\|,$$
$$\left|f(y) - f(x) - \langle G(x), y - x\rangle\right| \leq K\|y - x\|^2$$
whenever $x, y \in E$, $\|y - x\| \geq d$.

(ii) There exists a concave modulus $\omega \in \mathcal{M}$ such that f satisfies condition (WG_ω) *with G on E.*

Proof. (ii)⇒(i) Let $M > 0$ be the constant from condition (WG_ω). The function f clearly satisfies condition (W_G) (see (2)). Now let $d > 0$ be given. Then $\omega(t) = \omega\bigl(\frac{t}{d}d\bigr) \leq \frac{t}{d}\omega(d)$ for $t \geq d$ by Fact 10. It follows that
$$\|G(y) - G(x)\| \leq M\omega(\|y - x\|) \leq M\frac{\omega(d)}{d}\|y - x\|,$$
$$\left|f(y) - f(x) - \langle G(x), y - x\rangle\right| \leq M\omega(\|y - x\|)\|y - x\| \leq M\frac{\omega(d)}{d}\|y - x\|^2$$
whenever $x, y \in E$, $\|y - x\| \geq d$.

(i)⇒(ii) For each $\delta \geq 0$ set
$$\alpha(\delta) = \sup_{\substack{x,y\in E\\ 0<\|y-x\|\leq\delta}} \max\left\{\|G(y) - G(x)\|, \frac{\left|f(y) - f(x) - \langle G(x), y - x\rangle\right|}{\|y - x\|}\right\}$$
if $\{(x, y) \in E \times E;\ 0 < \|y - x\| \leq \delta\} \neq \emptyset$ and $\alpha(\delta) = 0$ otherwise. Obviously, $\alpha\colon [0, +\infty) \to [0, +\infty]$ is non-decreasing and $\alpha(0) = 0$. Using (W_G) we easily see that $\lim_{\delta\to 0_+} \alpha(\delta) = 0$. In particular there exists $\delta_0 > 0$ such that $\alpha(\delta) < 1$ for each $0 < \delta \leq \delta_0$. Let $K > 0$ be the constant from (i) corresponding to $d = \delta_0$. Then
$$\|G(y) - G(x)\| \leq K\|y - x\| \leq K\delta \quad \text{and} \quad \frac{\left|f(y) - f(x) - \langle G(x), y - x\rangle\right|}{\|y - x\|} \leq K\|y - x\| \leq K\delta$$
whenever $x, y \in E$ and $\delta_0 \leq \|y - x\| \leq \delta$, and so $\alpha(\delta) \leq \max\{1, K\delta\}$ for each $\delta \geq 0$. Hence $\alpha \in \mathcal{M}$ and by Lemma 12(i) there exists a concave modulus $\omega \in \mathcal{M}$ such that $\alpha \leq \omega$. Now if arbitrary $u, v \in E$ satisfying $u \neq v$ are given, then $\max\bigl\{\|G(v) - G(u)\|, \frac{|f(v)-f(u)-\langle G(u),v-u\rangle|}{\|v-u\|}\bigr\} \leq \alpha(\|v - u\|) \leq \omega(\|v - u\|)$ and so f satisfies (WG_ω) on E with G and $M = 1$. ∎

Lemma 56. *Let X be a normed linear space, let f be a function on $E \subset X$, and let $G\colon E \to X^*$. Then the following statements are equivalent:*

(i) f is Lipschitz, G is bounded, and f satisfies condition (W_G)*.*

(ii) There exists a bounded concave modulus $\omega \in \mathcal{M}$ such that f satisfies condition (WG_ω) *with G on E.*

Proof. (ii)⇒(i) The function f clearly satisfies condition (W_G) (see (2)). Further, suppose that f satisfies condition (WG_ω) on E with G and some $M > 0$. Let $K \geq 0$ be such that $\omega(t) \leq K$ for $t \geq 0$. Fix $x_0 \in E$ and set $L = \|G(x_0)\| + MK$. Then $\|G(x)\| \leq \|G(x_0)\| + \|G(x) - G(x_0)\| \leq \|G(x_0)\| + M\omega(\|x - x_0\|) \leq L$ for any $x \in E$. Finally,

$$|f(y)-f(x)| \leq \big|f(y)-f(x)-\langle G(x), y-x\rangle\big| + |\langle G(x), y-x\rangle| \leq M\omega(\|y-x\|)\|y-x\| + L\|y-x\| \leq (KM+L)\|y-x\|$$

for any $x, y \in E$.

(i)⇒(ii) Let $L > 0$ be such that f is L-Lipschitz and $\|G(x)\| \leq L$ for each $x \in E$. Then

$$\|G(y) - G(x)\| \leq 2L \quad \text{and} \quad \big|f(y) - f(x) - \langle G(x), y-x\rangle\big| \leq 2L\|y-x\| \tag{4}$$

whenever $x, y \in E$. So, if $d > 0$ and $x, y \in E$ are such that $\|y - x\| \geq d$, then $\|G(y) - G(x)\| \leq \frac{2L}{d}\|y - x\|$ and $\big|f(y) - f(x) - \langle G(x), y-x\rangle\big| \leq \frac{2L}{d}\|y-x\|^2$. Therefore by Lemma 55 there is a concave $\widetilde{\omega} \in \mathcal{M}$ such that f satisfies condition ($\mathrm{WG}_{\widetilde{\omega}}$) on E with G and a constant $M > 0$. Using this fact and (4) we obtain that f satisfies condition (WG_ω) on E with G and the constant M, where $\omega = \min\{\widetilde{\omega}, \frac{2L}{M}\}$. □

Theorem 57. *Let X be a normed linear space, $E \subset X$, and let f be a real function on E. Consider the following statements:*

(i) f can be extended to a function $F \in C^{1,+}(X)$.

(ii) There exists G such that f satisfies condition (W_G) and for each $d > 0$ there exists $K > 0$ such that

$$\|G(y) - G(x)\| \leq K\|y - x\|,$$
$$\big|f(y) - f(x) - \langle G(x), y-x\rangle\big| \leq K\|y-x\|^2$$

whenever $x, y \in E$, $\|y - x\| \geq d$.

Then (i)⇒(ii). If X is a super-reflexive Banach space that has an equivalent norm with modulus of smoothness of power type 2, then both statements are equivalent and moreover if (ii) holds, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.

Proof. (i)⇒(ii) By Fact 9, $\omega_{DF} \in \mathcal{M}$ is sub-additive and so by Lemma 12(iii) there is a concave $\omega \in \mathcal{M}$ such that $\omega_{DF} \leq \omega$. Fact 26 used on F implies that f satisfies condition (WG_ω) on E with $G = DF\restriction_E$ and we may apply Lemma 55.

(ii)⇒(i) By Lemma 55 there exists a concave modulus $\omega \in \mathcal{M}$ such that f satisfies condition (WG_ω) with G on E. By Theorem 2 the function f can be extended to a function $F \in C^{1,\omega}(X) \subset C^{1,+}(X)$ such that $DF(x) = G(x)$ for each $x \in E$. □

Remark 58. We do not know whether the assumption on X for (ii)⇒(i) in the preceding theorem (i.e. the existence of an equivalent norm with modulus of smoothness of power type 2) can be relaxed. However, the super-reflexivity of X is necessary: An extension of the function defined by $f = 0$ on $X \setminus B_X$ and $f(0) = 1$ (which satisfies (ii) with $G = 0$) produces a $C^{1,+}$-smooth bump, so we can use [DGZ, Theorem V.3.2].

The following theorem is an extended version of Theorem 6.

Theorem 59. *Let X be a super-reflexive Banach space, $E \subset X$, and let f be a real function on E. Consider the following statements:*

(i) f is bounded and satisfies condition (W_G) for some bounded G.

(ii) f is Lipschitz and satisfies condition (W_G) for some bounded G.

(iii) f can be extended to a Lipschitz function $F \in C^{1,+}(X)$.

(iv) f can be extended to a function $F \in C^{1,+}(X)$.

Then (i)⇒(ii)⇔(iii)⇒(iv). If E is bounded, then all four statements are equivalent.

Moreover, if (i) or (ii) holds, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.

For the most important implication (ii)⇒(iii) the assumption on X cannot be relaxed, see Remark 69. Cf. also Remark 71.

Proof. (i)⇒(ii) Let $K \geq 0$ be such that $\|G(x)\| \leq K$ and $|f(x)| \leq K$ for each $x \in E$. Condition (W_G) implies that there is $\delta > 0$ such that $\big|f(y) - f(x) - \langle G(x), y-x\rangle\big| \leq \|y - x\|$ whenever $x, y \in E$ are such that $\|y - x\| < \delta$. Now let $x, y \in E$. If $\|y - x\| < \delta$, then

$$|f(y) - f(x)| \leq \big|f(y) - f(x) - \langle G(x), y-x\rangle\big| + \big|\langle G(x), y-x\rangle\big| \leq \|y-x\| + K\|y-x\|.$$

On the other hand, if $\|y - x\| \geq \delta$, then $|f(y) - f(x)| \leq 2K = \frac{2K}{\delta}\delta \leq \frac{2K}{\delta}\|y - x\|$. Altogether, f is $\max\{1 + K, \frac{2K}{\delta}\}$-Lipschitz.

(ii)⇒(iii) By Lemma 56 there exists a bounded concave modulus $\omega \in \mathcal{M}$ such that f satisfies condition (WG_ω) with G on E. By Theorem 36 there is an equivalent norm $|||\cdot|||$ on X that has modulus of smoothness of power type $1 + \alpha$ for some $0 < \alpha \leq 1$. Lemma 27(iv) implies that f satisfies (WG_ω) in $(X, |||\cdot|||)$ with the same mapping G. Denoting $\varphi(t) = \int_0^t \omega(s)\,\mathrm{d}s$ for $t \geq 0$ and $\nu(x) = \varphi(|||x|||)$ for $x \in X$, Lemma 39(i) gives that $\nu \in C^{1,\sigma}_{|||\cdot|||}(X)$ for some bounded $\sigma \in \mathcal{M}$. By Proposition 42 used on the space $(X, |||\cdot|||)$ there is $F \in C^{1,\sigma}_{|||\cdot|||}(X)$ which is an extension of f with $DF = G$ on E. So $F \in C^{1,\sigma}_{\|\cdot\|}(X) \subset C^{1,+}(X)$ by Lemma 27(ii). Finally, since σ is bounded, DF is bounded as well, and therefore F is Lipschitz.

(iii)⇒(ii) By Fact 9, $\omega_{DF} \in \mathcal{M}$, and so F satisfies condition (WG$_{\omega_{DF}}$) with $G_1 = DF$ on X by Fact 26. Thus f satisfies condition (WG$_{\omega_{DF}}$) on E with $G = DF\restriction_E$ which implies that f satisfies condition (W$_G$) by (2). Moreover, DF is bounded as F is Lipschitz, and so G is bounded and clearly $f = F\restriction_E$ is Lipschitz.

(iii)⇒(iv) is trivial.

Now suppose that E is bounded and let us prove (iv)⇒(i). By Fact 9, $\omega_{DF} \in \mathcal{M}$, and so F satisfies condition (WG$_{\omega_{DF}}$) on X by Fact 26. Thus f satisfies condition (WG$_{\omega_{DF}}$) on E with $G = DF\restriction_E$ which implies that f satisfies condition (W$_G$) by (2). Further, let $B \subset X$ be a ball with $E \subset B$. Then DF is bounded on B, as it is uniformly continuous, and so F is Lipschitz on B and hence bounded on B. Consequently, both G and f are bounded.

□

Remark 60. In connection with Theorems 57 and 59 note that condition (W$_G$) (even with $G = 0$) alone does not imply the existence of a $C^{1,+}$-smooth extension even on $\mathbb{R}$. Indeed, let $E = \mathbb{N} \subset \mathbb{R}$ and let $f(2n) = 0$, $f(2n-1) = n$, and $G(n) = 0$ for $n \in \mathbb{N}$. Then f clearly satisfies (W$_G$). However, f cannot be extended to $F \in C^{1,+}(\mathbb{R})$: Suppose the contrary. Then for each $n \in \mathbb{N}$ the Mean value theorem implies the existence of $\zeta_n \in (2n-1, 2n)$ and $\xi_n \in (2n, 2n+1)$ such that $F'(\zeta_n) = -n$ and $F'(\xi_n) = n+1$. But $2n+1 = F'(\xi_n) - F'(\zeta_n) \le \omega_{F'}(2)$ for each $n \in \mathbb{N}$, so $\omega_{F'}(2) = +\infty$, a contradiction with Fact 9 (since moduli in $\mathcal{M}$ are finite).

Corollary 61. *Let X be a super-reflexive Banach space, $c \ge 1$, and let $U \subset X$ be an arbitrary union of uniformly separated open c-quasiconvex sets. If $f \in C^{1,+}(U)$ is Lipschitz, then it can be extended to a Lipschitz function $F \in C^{1,+}(X)$.*

The assumption on X cannot be relaxed even in the most interesting case when we extend from one open bounded quasiconvex set, see Remark 69.

Proof. Let $\delta_0 > 0$ be such that $U = \bigcup_{\gamma\in\Gamma} U_\gamma$, where each U_γ is an open c-quasiconvex set and $\operatorname{dist}(U_\alpha, U_\beta) \ge \delta_0$ for $\alpha, \beta \in \Gamma$, $\alpha \ne \beta$. Denote $f_\gamma = f\restriction_{U_\gamma}$. Given $\gamma \in \Gamma$, Lemma 21 implies that $\omega_{Df_\gamma} \in \mathcal{M}$ and Lemma 46 (with $\omega = \omega_{Df_\gamma}$ and $K = 1$) then implies that f satisfies (WG$_{\omega_{Df_\gamma}}$) on U_γ with $G = Df_\gamma$ and $M = 4c^3$. We claim that f satisfies condition (W$_{Df}$). Indeed, Df is uniformly continuous by the assumption. Further, let $\varepsilon > 0$. There is $0 < \delta < \delta_0$ such that $\omega_{Df}(\delta) \le \frac{\varepsilon}{4c^3}$. Now if $x, y \in U$, $\|y-x\| < \delta$, then there is $\gamma \in \Gamma$ such that $x, y \in U_\gamma$ and so $\left|f(y)-f(x)-\langle Df(x), y-x\rangle\right| = \left|f(y)-f(x)-\langle Df_\gamma(x), y-x\rangle\right| \le 4c^3\|y-x\|\omega_{Df_\gamma}(\|y-x\|) \le 4c^3\|y-x\|\omega_{Df}(\|y-x\|) \le \varepsilon\|y-x\|$.

Moreover, Df is bounded since f is Lipschitz. Therefore f can be extended to a Lipschitz function $F \in C^{1,+}(X)$ by Theorem 59, (ii)⇒(iii).

□

Corollary 62. *Let X be a super-reflexive Banach space, $U \subset X$ an open quasiconvex set, and $f \in C^{1,+}(U)$. Suppose that at least one of the following two conditions is satisfied:*

(a) X has an equivalent norm with modulus of smoothness of power type 2,

(b) U is bounded.

Then f can be extended to a function $F \in C^{1,+}(X)$.

For the case (b) the assumption of super-reflexivity of X cannot be relaxed, see Remark 69.

Proof. Suppose that X has an equivalent norm with modulus of smoothness of power type 2. Lemma 21 implies that there exists a concave $\omega \in \mathcal{M}$ such that $f \in C^{1,\omega}(U)$. Lemma 46 then implies that f satisfies (WG$_\omega$) on U. Therefore f can be extended to a function $F \in C^{1,+}(X)$ by Theorem 2.

Now suppose that U is bounded. Lemma 21 implies that $\omega_{Df} \in \mathcal{M}$ and so Df is bounded on U. Therefore f is Lipschitz (Proposition 22) and we may use Corollary 61.

□

Remark 63. The assumption of quasiconvexity in Corollaries 61 and 62 (and in Corollary 67 below) cannot be dropped even in the case when U is a Jordan domain. To this end, consider the set

$$U = U\big((0,0), 1\big) \setminus \big\{(x,y);\ x \ge 0, |y| \le x^2\big\} \subset \mathbb{R}^2$$

(which coincides with the set G_2 from [BB, p.136, Figure 2.1]) and define $f : U \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y < 0, \\ x^2 & \text{if } x \ge 0 \text{ and } y \ge 0. \end{cases}$$

Then it is easy to check that $f \in C^{1,\frac{1}{2}}(U) \subset C^{1,+}(U)$ and that f is Lipschitz. On the other hand, f cannot be extended to a function $F \in C^1(\mathbb{R}^2)$. Indeed, suppose that F is a such extension. Then for all sufficiently small $x > 0$ the Mean value theorem implies that for some point (x, ξ_x) with $-x^2 \le \xi_x \le x^2$ the equality

$$\frac{\partial F}{\partial y}(x, \xi_x) = \frac{F(x,x^2) - F(x,-x^2)}{2x^2} = \frac{1}{2}$$

holds, which contradicts the fact that $\frac{\partial F}{\partial y}(0,0) = 0$. In particular, $U = G_2$ is not (C,ω)-convex for $\omega(t) = t^{\frac{1}{2}}$, contrary to what is stated in [BB, p. 136], since otherwise f would have an extension $F \in C^1(X)$ by [BB, Theorem 2.64]. (Cf. also Remark 48.)

Remark 64. We do not know whether conditions (a) and (b) in Corollary 62 can be disposed of; see Corollary 67 for a weaker result. Note that they can be disposed of if f is Lipschitz (Corollary 61).

Remark 65. We do not know whether it is possible to extend $C^{1,+}$-smooth functions similarly as in Corollaries 47, 53, 61, and 62 from open bounded *convex* sets (resp. open balls) in some space that is not super-reflexive. An example by Petr Hájek from [D'H] implies that it is not possible in c_0 (resp. in a space isomorphic to c_0): There are an open absolutely convex bounded set $U \subset c_0$ and functions $\Phi_\beta \in C^\infty(U)$, $\beta > 1$, with all derivatives bounded and Lipschitz, such that Φ_β cannot be extended to a function from $C^{1,+}(\beta U)$. This formulation follows rather easily from [HJ, remark after Proposition 6.33]. (For a more striking example on a different space see also [HJ, Theorem 6.34].)

6. THE $C_{\mathrm{B}}^{1,+}$ AND $C_{\mathrm{loc}}^{1,+}$ CASES

The following theorem is an extended version of Theorem 4.

Theorem 66. *Let X be a super-reflexive Banach space, $E \subset X$, and let f be a real function on E. Then the following statements are equivalent:*

(i) f can be extended to a real function $F \in C_{\mathrm{B}}^{1,+}(X)$.
(ii) f satisfies condition $(\widetilde{\mathrm{W}})$ on E.
(iii) For each bounded $A \subset E$ the function f is bounded on A and there exists a bounded $G_A \colon A \to X^*$ such that $f \restriction_A$ satisfies condition (W_{G_A}).

Moreover, if $(\widetilde{\mathrm{W}})$ is satisfied with G, then F can be found such that $DF(x) = G(x)$ for each $x \in E$.

For the most important implication (iii)⇒(i) the assumption on X cannot be relaxed, see Remark 69.

Proof. (i)⇒(ii) We will show that f satisfies condition $(\widetilde{\mathrm{W}})$ on E with $G = DF \restriction_E$. To this end consider an arbitrary bounded $B \subset E$ and choose an open convex bounded $U \supset B$. Denote $H = F \restriction_U$. By Fact 9 and Lemma 12(iii) there is a concave $\widetilde{\omega} \in \mathcal{M}$ such that $H \in C^{1,\widetilde{\omega}}(U)$. Since U is bounded, $H \in C^{1,\omega}(U)$ for $\omega(t) = \min\{\widetilde{\omega}(t), \widetilde{\omega}(\operatorname{diam} U)\}$. As H satisfies (WG_ω) with $G_1 = DF \restriction_U$ on U by Fact 26, Lemma 56 implies that H satisfies (W_{G_1}), G_1 is bounded, and H is Lipschitz. Then H is bounded, since U is bounded. Consequently, $f \restriction_B$ satisfies $(\mathrm{W}_{G\restriction_B})$ and both G and f are bounded on B. Since B was arbitrary, the claim clearly follows.

(ii)⇒(iii) is trivial.

Finally, we will prove (iii)⇒(i) and (ii)⇒(i) with the "moreover" part both at once. By Theorem 36 there is an equivalent norm ν on X with modulus of smoothness of power type $1 + \alpha$ for some $0 < \alpha \le 1$. Then $\nu^{1+\alpha} \in C_\nu^{1,\alpha}(X)$ by [DGZ, proof of Lemma IV.5.9] combined with Lemma 38, and so $\nu^{1+\alpha} \in C^{1,\alpha}(X)$ by Lemma 27(ii). Let $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ be a locally finite C^2-smooth partition of unity on $\mathbb{R}$ subordinated to the covering $\{(-n, n);\ n \in \mathbb{N}\}$ (for the construction of such partition see e.g. [Sp, Theorem 3-11]). For each $\gamma \in \Gamma$ set $\psi_\gamma = \varphi_\gamma \circ \nu^{1+\alpha}$. Since the support of each φ_γ is compact, it follows that φ_γ' is bounded and also φ_γ'' is bounded, which implies that φ_γ' is Lipschitz. Since $D\nu^{1+\alpha}$ is α-Hölder on X, it follows that $D\nu^{1+\alpha}$ is bounded on bounded sets and consequently $\nu^{1+\alpha}$ is Lipschitz on bounded sets ([HJ, Proposition 1.71]). Then $\nu^{1+\alpha}$ is α-Hölder on bounded sets, since for each $r > 0$ there is $C > 0$ such that $Ct^\alpha \ge t$ for $t \in [0, r]$. By [HJ, Proposition 1.128(ii)] used with $k = 1$ and $\omega(t) = Kt^\alpha$ for a sufficiently large K it follows that $\psi_\gamma \in C^{1,\alpha}(U)$ for every bounded open $U \subset X$.

For each $\gamma \in \Gamma$ the set $\operatorname{supp}_{\mathrm{o}} \varphi_\gamma$ is bounded and so also $\operatorname{supp}_{\mathrm{o}} \psi_\gamma$ and $\operatorname{supp} \psi_\gamma$ are bounded. Now since for each $\gamma \in \Gamma$ the function $f \restriction_{E\cap\operatorname{supp}\psi_\gamma}$ is bounded and satisfies condition (W_H) for some bounded $H \colon E \cap \operatorname{supp} \psi_\gamma \to X^*$ (by (iii) with $A = E \cap \operatorname{supp} \psi_\gamma$), Theorem 59 implies that there exists $F_\gamma \in C^{1,+}(X)$ which is an extension of $f \restriction_{E\cap\operatorname{supp}\psi_\gamma}$. In case that (ii) holds, we have a corresponding $G \colon E \to X^*$ and we can take $H = G \restriction_{E\cap\operatorname{supp}\psi_\gamma}$, so by Theorem 59 we may additionally assume that $DF_\gamma(x) = G(x)$ for each $x \in E \cap \operatorname{supp} \psi_\gamma$. Set $F = \sum_{\gamma\in\Gamma} \psi_\gamma F_\gamma$ and note that the sum is in fact finite on bounded sets: Let $r > 0$. Since $\{\operatorname{supp}_{\mathrm{o}} \varphi_\gamma\}$ is a locally-finite system, it follows that there is $\Gamma_r \subset \Gamma$ finite such that $\varphi_\gamma = 0$ on the compact interval $[0, r^{1+\alpha}]$ for $\gamma \notin \Gamma_r$. Hence $\psi_\gamma = 0$ (and so also $\psi_\gamma F_\gamma = 0$) on $U_\nu(0, r)$ for $\gamma \notin \Gamma_r$, which implies that $F(x) = \sum_{\gamma\in\Gamma_r} \psi_\gamma(x) F_\gamma(x)$ for $x \in U_\nu(0, r)$.

Further, note that $\psi_\gamma F_\gamma \in C_{\mathrm{B}}^{1,+}(X)$ for each $\gamma \in \Gamma$. Indeed, if $V \subset X$ is an open ball, then ψ_γ, $D\psi_\gamma$, and DF_γ are bounded on V. It follows that ψ_γ and F_γ are Lipschitz on V, thus F_γ is bounded on V, and in turn all four mappings ψ_γ, $D\psi_\gamma$, F_γ, and DF_γ are uniformly continuous on V with the same modulus (Fact 11). It follows that $\psi_\gamma F_\gamma \in C^{1,+}(V)$ ([HJ, Proposition 1.129]). Consequently, $F \in C_{\mathrm{B}}^{1,+}(X)$, as the sum in its definition is finite on bounded sets.

To see that F is an extension of f let $x \in E$. Let $r > 0$ be such that $x \in U_\nu(0, r)$ and let Γ_r be as above. Put $\Lambda = \{\gamma \in \Gamma_r;\ x \in \operatorname{supp} \psi_\gamma\}$. Then $\psi_\gamma(x) = 0$ for $\gamma \in \Gamma \setminus \Lambda$ and $F_\gamma(x) = f \restriction_{E\cap\operatorname{supp}\psi_\gamma}(x) = f(x)$ whenever $\gamma \in \Lambda$. Hence

$$F(x) = \sum_{\gamma\in\Lambda} \psi_\gamma(x) F_\gamma(x) = \sum_{\gamma\in\Lambda} \psi_\gamma(x) f(x) = f(x) \sum_{\gamma\in\Lambda} \psi_\gamma(x) = f(x) \sum_{\gamma\in\Gamma} \psi_\gamma(x) = f(x).$$

Finally, assuming that (ii) holds we show that $DF(x) = G(x)$. Since $\sum_{\gamma\in\Gamma_r} \psi_\gamma(y) = 1$ for any $y \in U_\nu(0, r)$, it follows that $\sum_{\gamma\in\Gamma} D\psi_\gamma(x) = \sum_{\gamma\in\Gamma_r} D\psi_\gamma(x) = 0$. Moreover, $D\psi_\gamma(x) = 0$ whenever $\gamma \in \Gamma_r \setminus \Lambda$, as the set $X \setminus \operatorname{supp} \psi_\gamma$ is open, and

$DF_\gamma(x) = G(x)$ for $\gamma \in \Lambda$. Hence

$$\begin{aligned} DF(x) &= \sum_{\gamma\in\Gamma_r} \psi_\gamma(x)DF_\gamma(x) + F_\gamma(x)D\psi_\gamma(x) = \sum_{\gamma\in\Lambda} \psi_\gamma(x)DF_\gamma(x) + F_\gamma(x)D\psi_\gamma(x) \\ &= \sum_{\gamma\in\Lambda} \psi_\gamma(x)G(x) + f(x)D\psi_\gamma(x) = G(x)\sum_{\gamma\in\Gamma}\psi_\gamma(x) + f(x)\sum_{\gamma\in\Gamma} D\psi_\gamma(x) = G(x). \end{aligned}$$

□

Corollary 67. *Let X be a super-reflexive Banach space, $U \subset X$ an open quasiconvex set, and $f \in C^{1,+}(U)$. Then f can be extended to a function $F \in C^{1,+}_{\mathrm{B}}(X)$.*

The assumption on X cannot be relaxed, see Remark 69.

Proof. Lemma 21 implies that there exists $\omega \in \mathcal{M}$ such that $f \in C^{1,\omega}(U)$. Lemma 46 then implies that f satisfies (WG_ω) on U with $G = Df$ and some $M > 0$. Now let $A \subset U$ be an arbitrary bounded subset. Fixing some $x_0 \in A$, the inequalities in (WG_ω) imply that $\|G(y)\| \le \|G(x_0)\| + M\omega(\operatorname{diam} A)$ and $|f(y)| \le |f(x_0)| + \|G(x_0)\| \operatorname{diam} A + M\omega(\operatorname{diam} A)\operatorname{diam} A$ for each $y \in A$, and so both f and G are bounded on A. Also, $f\restriction_A$ satisfies $(\mathrm{W}_{G\restriction_A})$ by (2). Therefore f can be extended to a function $F \in C^{1,+}_{\mathrm{B}}(X)$ by Theorem 66, (iii)⇒(i).

□

At least for certain quasiconvex sets it is possible to relax the assumption on f in the previous corollary:

Corollary 68. *Let X be a super-reflexive Banach space, let $U \subset X$ be an open set that is a bi-Lipschitz image of a convex subset of a normed linear space, and let $f \in C^{1,+}_{\mathrm{B}}(U)$. Then f can be extended to a function $F \in C^{1,+}_{\mathrm{B}}(X)$.*

Proof. Let $\Phi\colon U \to V$ be a bi-Lipschitz mapping onto, where V is a convex subset of some normed linear space. Let $A \subset U$ be an arbitrary bounded subset. Then $\Phi(A)$ is bounded and so there is $R > 0$ such that $\Phi(A) \subset U(0,R)$. The set $V \cap U(0,R)$ is convex and relatively open in V, and so $W = \Phi^{-1}(V \cap U(0,R))$ is open and it is a bi-Lipschitz image of a bounded convex set, hence it is quasiconvex (Remark 19) and bounded. Moreover, $A \subset W$.

Since $f\restriction_W \in C^{1,+}(W)$, Corollary 62 combined with Theorem 59, (iv)⇒(i) (used for the bounded $E = W$), implies that $f\restriction_W$ is bounded and satisfies condition (W_G) for some bounded G. Thus f is bounded on A and $f\restriction_A$ satisfies condition $(\mathrm{W}_{G\restriction_A})$ for some bounded G. Therefore f can be extended to a function $F \in C^{1,+}_{\mathrm{B}}(X)$ by Theorem 66.

□

Remark 69. In Remark 54 we showed that the assumptions on X in Theorems 2 and 52 cannot be relaxed. Now we show that

the assumption on X in Corollaries 47, 53, 61, 62"(b)", 67, and Theorems 59 and 66 cannot be relaxed. (5)

(a) We start with a construction of a bounded open quasiconvex set U in an arbitrary normed linear space X with $\dim X \ge 2$ and a $C^{1,1}$-smooth function f on it. Fix $e \in S_X$. Set $C = \bigcup_{t>0} tU(e, \frac{1}{4})$ and $U = (U(0,2) \setminus B_X) \cup (U(0,2) \cap C)$. We claim that U is 5-quasiconvex. Since $U(0,2) \setminus B_X$ is 2-quasiconvex by Example 20 and C is convex, it suffices to consider $x \in U(0,2) \setminus B_X$ and $y \in C \cap B_X$. Set $z = \|x\|\frac{y}{\|y\|} \in U(0,2) \setminus B_X$. Then $\|z - y\| = \|z\| - \|y\| = \|x\| - \|y\| \le \|x - y\|$ and $\|x - z\| \le \|x - y\| + \|y - z\| \le 2\|x - y\|$. By the 2-quasiconvexity of $U(0,2) \setminus B_X$ there is a continuous rectifiable curve γ joining x and z in U such that $\operatorname{len}\gamma \le 2\|x - z\|$. By concatenating γ with the segment $[z, y] \subset U$ we obtain a continuous rectifiable curve joining x and y in U whose length is at most $2\|x - z\| + \|z - y\| \le 5\|x - y\|$.

Next, by the Hahn-Banach theorem there is $g \in S_{X^*}$ such that $g(e) = 1$. Note that

$$\|x\| < \frac{5}{6} \text{ whenever } x \in C \text{ and } g(x) \le \frac{1}{2}. \tag{6}$$

Indeed, there is $t > 0$ such that $x \in U(te, \frac{1}{4}t)$. Then $t = g(te - x) + g(x) \le \|te - x\| + \frac{1}{2} < \frac{1}{4}t + \frac{1}{2}$, which implies that $t < \frac{2}{3}$, and consequently $\|x\| \le \|x - te\| + \|te\| < \frac{5}{4}t < \frac{5}{6}$.

Now let $\varphi \in C^{1,1}(\mathbb{R})$ be such that $\varphi = 0$ on $[\frac{1}{2}, +\infty)$, $\varphi(\frac{1}{4}) > 0$, and $|\varphi'|$ is bounded by a constant $M > 0$. Define $f\colon U \to \mathbb{R}$ by $f = \varphi \circ g$ on $C \cap U$ and $f = 0$ on $U \setminus B_X$. Notice that f is well-defined, since $g > \frac{1}{2}$ on $C \setminus B_X$ by (6). The function f is clearly $C^{1,1}$-smooth on $C \cap U$ and also on $U \setminus \{x \in C;\ g(x) \le \frac{1}{2}\}$ ($Df = 0$ on this set). Further, if $x \in C \cap U$ with $g(x) \le \frac{1}{2}$ and $y \in U \setminus C$, then $\|Df(x) - Df(y)\| = \|Df(x)\| = \|D\varphi(g(x)) \circ g\| \le |\varphi'(g(x))| \cdot \|g\| \le M$. On the other hand, $\|x - y\| \ge \|y\| - \|x\| > 1 - \frac{5}{6} = \frac{1}{6}$ (use (6)). It follows that $\|Df(x) - Df(y)\| \le M = 6M\frac{1}{6} < 6M\|x - y\|$. Consequently, $f \in C^{1,1}(U)$.

Note that $f \in C^{1,1}(U) \subset C^{1,\alpha}(U) \subset C^{1,+}(U)$ for any $0 < \alpha \le 1$ (the first inclusion follows since U is bounded) and that f is Lipschitz (Proposition 22). Let S be one of the spaces $C^{1,\alpha}(X)$, $C^{1,+}(X)$, $C^{1,+}_{\mathrm{B}}(X)$. Assume that we can extend f to a function $F \in S$. Let $H = \chi_{U(0,2)} \cdot F$. Since $H = 0$ on $X \setminus B_X$ and also $H = F = 0$ on $U(0,2) \setminus B_X$, it is easy to check that $H \in S$. Moreover, H is a bump function. In particular, $H \in C^{1,+}(X)$ even in the case $S = C^{1,+}_{\mathrm{B}}(X)$. It follows that X is super-reflexive and in case that $S = C^{1,\alpha}(X)$ it has an equivalent norm with modulus of smoothness of power type $1 + \alpha$ ([DGZ, Theorem V.3.2]).

(b) To prove (5) assume that Corollary 47, resp. 53, resp. 61, resp. 62(b), resp. 67 holds for some X which does not satisfy the respective assumptions. Now we apply (a) with $S = C^{1,1}(X) = C^{1,\omega}(X)$ for $\omega(t) = t$ for Corollary 47, resp. $S = C^{1,\alpha}(X)$ for Corollary 53, resp. $S = C^{1,+}(X)$ for Corollaries 61, 62, resp. $S = C^{1,+}_{\mathrm{B}}(X)$ for Corollary 67, which leads to a contradiction. Further, since the assumptions on X in Corollary 61, resp. 67, cannot be relaxed, it follows that the assumptions on X for Theorem 59, (ii)⇒(iii), resp. Theorem 66, (iii)⇒(i) cannot be relaxed as well, since Corollary 61, resp. 67, is a direct consequence of the just mentioned respective implications.

Using Theorem 59 and another partition of unity we obtain the following result.

Theorem 70. *Let X be a super-reflexive Banach space, $U \subset X$ an open set, $A \subset U$ relatively closed in U, and $f : A \to \mathbb{R}$. Then f can be extended to a function $F \in C^{1,+}_{\mathrm{loc}}(U)$ if and only if f is locally bounded and for each $x \in A$ there exist $\delta_x > 0$ and a bounded $G_x : A \cap U(x, \delta_x) \to X^*$ such that $f\restriction_{A\cap U(x,\delta_x)}$ satisfies condition (W_{G_x}).*

Proof. ⇒ F is clearly locally bounded and hence so is f. Choose any $x \in A$. There is $\delta_x > 0$ such that DF is uniformly continuous on $U(x, \delta_x) \subset U$. Set $H = F\restriction_{U(x,\delta_x)}$. Since $\omega_{DH} \in \mathcal{M}$ by Fact 9, DH is bounded and Fact 26 implies that H satisfies condition $(\mathrm{WG}_{\omega_{DH}})$ on $U(x, \delta_x)$ with $G = DH$. Therefore H clearly satisfies condition (W_{DH}) (see (2)). Consequently, $f\restriction_{A\cap U(x,\delta_x)}$ satisfies condition $(\mathrm{W}_{DH\restriction_{A\cap U(x,\delta_x)}})$.

⇐ To every $x \in A$ we assign $\delta_x > 0$ as in the assumption (of the current implication) for which moreover $U(x, \delta_x) \subset U$ and f is bounded on $A \cap U(x, \delta_x)$. Let $\{\varphi_\alpha\}_{\alpha\in\Lambda'}$ be a locally finite $C^{1,+}$-smooth partition of unity on U subordinated to the open covering $\{U(x, \delta_x)\}_{x\in A} \cup \{U \setminus A\}$. (The existence of such partition follows from Theorem 36 and [Kr2, Proposition 3.7], whose proof is a simple modification of the proof of [HJ, Theorem 7.56]. Another possibility is described in Remark 54.) Set $\Lambda = \{\alpha \in \Lambda';\ \mathrm{supp}_{\mathrm{o}}\,\varphi_\alpha \cap A \neq \emptyset\}$. For each $\alpha \in \Lambda$ let $x_\alpha \in A$ be such that $\mathrm{supp}_{\mathrm{o}}\,\varphi_\alpha \subset U(x_\alpha, \delta_{x_\alpha})$. Denote $U_\alpha = U(x_\alpha, \delta_{x_\alpha})$. By Theorem 59, (i)⇒(iv), for each $\alpha \in \Lambda$ there is $F_\alpha \in C^{1,+}(X)$ which is an extension of $f\restriction_{A\cap U_\alpha}$. Now put $F = \sum_{\alpha\in\Lambda} \varphi_\alpha F_\alpha$.

First note that $\varphi_\alpha F_\alpha \in C^{1,+}_{\mathrm{loc}}(U)$ for each $\alpha \in \Lambda$. Indeed, given $x \in U$, by continuity there is $\delta > 0$ such that $V = U(x, \delta) \subset U$ and φ_α, $D\varphi_\alpha$, F_α, and DF_α are bounded on V. It follows that φ_α and F_α are Lipschitz on V, and in turn all four mappings are uniformly continuous on V with the same modulus (Fact 11). It follows that $\varphi_\alpha F_\alpha \in C^{1,+}(V)$ ([HJ, Proposition 1.129]). Consequently, $F \in C^{1,+}_{\mathrm{loc}}(U)$, as the sum in its definition is locally finite: for every $x \in U$ there is a neighbourhood W of x and $H \subset \Lambda$ finite such that $\varphi_\alpha F_\alpha = 0$ on W for $\alpha \in \Lambda \setminus H$, and so $F = \sum_{\alpha\in H} \varphi_\alpha F_\alpha$ on W.

Finally, to show that F is an extension of f suppose that $x \in A$ is given. Then $\varphi_\alpha(x) = 0$ for each $\alpha \in \Lambda' \setminus \Lambda$ and for each $\alpha \in \Lambda$ such that $x \notin U_\alpha$. Hence

$$F(x) = \sum_{\substack{\alpha\in\Lambda\\ x\in U_\alpha}} \varphi_\alpha(x) F_\alpha(x) = \sum_{\substack{\alpha\in\Lambda\\ x\in U_\alpha}} \varphi_\alpha(x) f(x) = f(x) \sum_{\alpha\in\Lambda'} \varphi_\alpha(x) = f(x).$$

□

Remark 71. The assumption on X in the previous theorem cannot be relaxed even in the case $U = X$. In fact we show more: Even a weaker version of (i)⇒(iv) in Theorem 59, namely

$$\text{if } E \text{ is closed, then (i) implies that } f \text{ can be extended to } F \in C^{1,+}_{\mathrm{loc}}(X) \tag{7}$$

holds only in super-reflexive spaces X. Note that the validity of Theorem 70 for some space X implies (7), since the “global” statement (i) in Theorem 59 is stronger than the condition in Theorem 70, which is a “localised” version of this statement.

Indeed, suppose that (7) holds for some space X. Then there exists a $C^{1,+}_{\mathrm{loc}}$-smooth bump on X (extend the function defined by $f = 0$ on $X \setminus U(0, 1)$ and $f(0) = 1$). We claim that X does not contain a subspace isomorphic to c_0 and so X is super-reflexive by [FWZ], see [HJ, Corollary 5.51]. Suppose to the contrary that X contains a subspace Y isomorphic to c_0. By the construction below there is a bounded function f on a closed subset $A \subset c_0$ that satisfies (W_G) with $G = 0$, where $G : A \to c_0^*$, and yet

$$f \text{ cannot be extended to a function on } c_0 \text{ that is } C^{1,+}\text{-smooth on a neighbourhood of } 0. \tag{8}$$

Identifying A with the corresponding subset of Y it is easy to see that f satisfies condition (W_H) in X with $H = 0$, where $H : A \subset Y \to X^*$. So by (7) the function f can be extended to $F \in C^{1,+}_{\mathrm{loc}}(X)$. Then $F\restriction_Y \in C^{1,+}_{\mathrm{loc}}(Y)$ is also an extension of f, which contradicts (8) (by Lemma 27(iii)).

We set $A = \{0\} \cup \left\{\frac{1}{2^n} e_k;\ n, k \in \mathbb{N}\right\}$ and define $f : A \to \mathbb{R}$ by $f(0) = 0$ and $f\left(\frac{1}{2^n} e_k\right) = \left(\frac{1}{2^n}\right)^2$. The set A is clearly closed. We claim that f satisfies (W_G) with $G = 0$: Given $\varepsilon > 0$ set $\delta = \frac{\varepsilon}{4}$. Now let $x, y \in A$, $0 < \|x - y\| < \delta$. We may assume that $x = \frac{1}{2^n} e_k$ for some $n, k \in \mathbb{N}$. If $y = 0$, then $|f(x) - f(y)| = \left(\frac{1}{2^n}\right)^2 = \|x - y\|^2 < \delta\|x - y\| < \varepsilon\|x - y\|$. Otherwise $y = \frac{1}{2^m} e_l$ for some $m, l \in \mathbb{N}$. We may assume without loss of generality that $m \geq n$. If $k = l$, then $\|x - y\| = \frac{1}{2^n} - \frac{1}{2^m} \geq \frac{1}{2^{n+1}}$, otherwise $\|x - y\| = \frac{1}{2^n} > \frac{1}{2^{n+1}}$. Therefore $|f(x) - f(y)| < \left(\frac{1}{2^n}\right)^2 \leq 4\|x - y\|^2 < 4\delta\|x - y\| = \varepsilon\|x - y\|$.

Now since $\lim_{k\to\infty} \frac{1}{2^n} e_k = 0$ weakly, it follows that f is not weakly sequentially continuous on any neighbourhood of 0. Thus (8) follows from results in [H], see [HJ, Theorem 6.30] and notice that clearly $\mathscr{C}_{\mathrm{wsC}} \subset \mathscr{C}_{\mathrm{wsc}}$ (cf. [HJ, pp. 137–138]).

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