SMOOTHNESS VIA DIRECTIONAL SMOOTHNESS AND MARCHAUD'S THEOREM IN BANACH SPACES

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ABSTRACT. Classical Marchaud's theorem (1927) asserts that if f is a bounded function on $[a, b]$, $k \in \mathbb{N}$, and the $(k+1)$th modulus of smoothness $\omega_{k+1}(f;t)$ is so small that $\eta(t) = \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s < +\infty$ for $t > 0$, then $f \in C^k\big((a,b)\big)$ and $f^{(k)}$ is uniformly continuous with modulus $c\eta$ for some $c > 0$ (i.e. in our terminology f is $C^{k,c\eta}$-smooth). Using a known version of the converse of Taylor theorem we easily deduce Marchaud's theorem for functions on certain open connected subsets of Banach spaces from the classical one-dimensional version. In the case of a bounded subset of $\mathbb{R}^n$ our result is more general than that of H. Johnen and K. Scherer (1973), which was proved by quite a different method. We also prove that if a locally bounded mapping between Banach spaces is $C^{k,\omega}$-smooth on every line, then it is $C^{k,c\omega}$-smooth for some $c > 0$.

1. INTRODUCTION

Classical Marchaud's theorem (1927) asserts that if f is a bounded function on $[a, b]$, $k \in \mathbb{N}$, and the $(k+1)$th modulus of smoothness $\omega_{k+1}(f;t)$ is so small that $\eta(t) = \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s < +\infty$ for $t > 0$, then $f \in C^k\big((a,b)\big)$ and $f^{(k)}$ is uniformly continuous with modulus $c\eta$ for some $c > 0$. Marchaud's theorem was generalised to functions defined on bounded LG domains in $\mathbb{R}^n$ in [JS] (it is an easy consequence of Theorems 1 and 2 of [JS]).

We will generalise Marchaud's theorem to mappings on certain open connected subsets of Banach spaces. Namely our proof works if the domain has the "uniform convex chain (UCC) property" (in particular, if it is convex and bounded or if it is convex and contains an unbounded cone). We show that our version of Marchaud's theorem (in $\mathbb{R}^n$) is more general than that of [JS] (which works with LG domains), see Proposition 21.

We deduce our version of Marchaud's theorem rather easily from the classical one-dimensional version and a recent ([J]) quantitative version of the Converse Taylor theorem. Both our proof and the proof of the result of [J] use only a little of analysis; they are essentially based on several non-trivial but well-known properties of polynomials in Banach spaces.

The rough strategy of our proof is the following: If f satisfies the assumptions of Marchaud's theorem in $U \subset X$, then its one-dimensional version implies that f is $C^{k,c_1\eta}$-smooth on all segments in U. For convex U we then show that this fact implies (see Proposition 12) that f is $C^{k,c_2\eta}$-smooth on U. This is done by verifying (using Theorem 9) that near each $a \in U$ the mapping f is well approximated by a polynomial of degree at most k and so the quantitative version of the Converse Taylor theorem of [J] implies that f is $C^{k,c_2\eta}$-smooth on U.

The above method gives indeed very easily Marchaud's theorem in higher dimensions under the assumption of continuity of f on convex domains, see Proposition 13. We believe that this simple proof may be interesting also for people working in the function spaces theory in $\mathbb{R}^n$. Moreover, it generalises to the domains with the UCC property quite easily, see Remark 14.

In the main section (Section 4) we prove a more general version (for vector-valued mappings on UCC domains) of Marchaud's theorem (Theorem 20) under much weaker assumptions (e.g. for locally bounded mappings). This generalisation needs some additional technical results, nevertheless we believe it is interesting from the point of view of differentiation theory in Banach spaces.

Theorem 9 also immediately implies Theorem 10 which is a new "directional version" of the Converse Taylor theorem and can be of some independent interest. Another new interesting result is a characterisation of $C^{k,c\omega}$-smoothness via the smoothness on one-dimensional affine subspaces in the domain (Theorem 19), which is analogous to the characterisation of polynomials or holomorphic mappings.

2. PRELIMINARIES

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}$ we denote by $\lceil x \rceil$ the ceiling of x, i.e. the unique number $k \in \mathbb{Z}$ satisfying $k - 1 < x \le k$. All vector spaces considered are real. We denote by $B(x,r)$, resp. $U(x,r)$ the closed, resp. open ball in a normed linear space centred at x with radius $r > 0$. By B_X we denote the closed unit ball of a normed linear space X, i.e. $B_X = B(0,1)$. By S_X we denote the unit sphere of a normed linear space X. Let X, Y be normed linear spaces and $n \in \mathbb{N}$. By $\mathcal{L}(^nX;Y)$ we denote the space of continuous n-linear mappings from X^n to Y with the norm $\|M\| = \sup_{x_1,\dots,x_n \in B_X} \|M(x_1,\dots,x_n)\|$. Recall that a continuous n-homogeneous polynomial from X to Y is a mapping P that is given by $P(x) = M(x,\dots,x)$, $x \in X$, for some $M \in \mathcal{L}(^nX;Y)$. By $\mathcal{P}(^nX;Y)$ we denote the space of continuous n-homogeneous polynomials from X to Y with the norm $\|P\| = \sup_{x \in B_X} \|P(x)\|$. By $\mathcal{P}^n(X;Y)$ we denote the space of continuous polynomials of degree at most n from X to Y with the

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norm $\|P\| = \sup_{x \in B_X} \|P(x)\|$. Of course, $\mathcal{P}^0(X;Y) = \mathcal{P}({}^0X;Y)$ is the space of constant mappings. The Polarisation formula (see e.g. [MO1, Statement 8, p. 62] or [HJ, Proposition 1.13]) implies the following fact:

Fact 1. *Let X, Y be normed linear spaces and let $n \in \mathbb{N}$. There is a mapping $I\colon \mathcal{P}({}^nX;Y) \to \mathcal{L}({}^nX;Y)$ that satisfies $P(h) = I(P)(h, \dots, h)$ which is a linear isomorphism into and $\|P\| \le \|I(P)\| \le \frac{n^n}{n!}\|P\|$.*

The mapping $f\colon X \to Y$ between topological spaces X and Y is said to be Baire measurable (or to have the Baire property) if $f^{-1}(G) \subset X$ has the Baire property for every $G \subset Y$ open, so each Borel measurable mapping is also Baire measurable.

Let X, Y be vector spaces, $U \subset X$, and $f\colon U \to Y$. We define the first difference by

$$\Delta^1 f(x;h) = f(x+h) - f(x)$$

for all $x \in U$ and $h \in X$ such that the right-hand side is defined. Further, we inductively define the differences of higher order by

$$\Delta^n f(x;h_1,\dots,h_n) = \Delta^{n-1} f(x+h_1;h_2,\dots,h_n) - \Delta^{n-1} f(x;h_2,\dots,h_n)$$

for all $x \in U$ and $h_1,\dots,h_n \in X$ such that the right-hand side is defined. We are also going to use the classical notation $\Delta_h^n f(x) = \Delta^n f(x;h,\dots,h)$. It is easy to check that

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh). \tag{1}$$

We will rely on the following fundamental result; it is a combination of [MO1, Satz I] and [MO2, Statement 12, p. 182], see also [HJ, Corollary 1.55]:

Theorem 2 ([MO1], [MO2])**.** *Let X be a Banach space, Y a normed linear space, $n \in \mathbb{N}_0$, and let $P\colon X \to Y$ be such that $\phi \circ P$ is Baire measurable for each $\phi \in Y^*$. Then $P \in \mathcal{P}^n(X;Y)$ if and only if P satisfies the (Fréchet) formula*

$$\Delta_h^{n+1} P(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x+kh) = 0 \tag{2}$$

for all $x, h \in X$.

We remark that the above characterisation holds also in incomplete spaces X provided that we assume that $\phi \circ P$ is continuous ([MO1, Satz I*], see also [HJ, Theorem 2.49, Fact 1.49]).

The following characterisation is an immediate consequence: A Baire measurable mapping $P\colon X \to Y$ is a continuous polynomial of degree at most n if and only if the restriction of P to each one-dimensional affine subspace of X is a polynomial of degree at most n; moreover, it suffices to test only the polynomiality of $\phi \circ P$, $\phi \in Y^*$. We note that a similar characterisation holds also for holomorphic mappings (of course in the case of complex Banach spaces). One of our main results below is that $C^{k,c\omega}$-smoothness can be characterised in the same way (Theorem 19, Theorem 17).

Let X, Y be normed linear spaces, $U \subset X$ open, $f\colon U \to Y$, and $x \in U$. By $Df(x)$ we denote the Fréchet derivative of f at x, and by $Df(x)[h]$ we denote the evaluation of this derivative in $h \in X$. Similarly we denote by $D^k f(x)$ the kth Fréchet derivative of f at x. By $d^k f(x)$ we denote the k-homogeneous polynomial corresponding to the symmetric k-linear mapping $D^k f(x)$, so $d^k f(x)[h] = D^k f(x)[h,\dots,h]$. For convenience we put $d^0 f = f$. Recall that we identify $D^k f(x)$ and $D(D^{k-1}f)(x)$ using the formula $D^k f(x)[h_1,\dots,h_k] = \bigl(D(D^{k-1}f)(x)[h_1]\bigr)(h_2,\dots,h_k)$.

We recall the following well-known easy facts:

Fact 3. *Let X be a normed linear space, $U \subset X$ open, $f\colon U \to \mathbb{R}$, $x \in U$, $h \in X$, and $k \in \mathbb{N}$. Further, let $g(t) = f(x+th)$. Then $d^k f(x+th)[h] = g^{(k)}(t)$ if the left-hand side exists.*

Fact 4. *Let X, Y, Z be normed linear spaces, $L \in \mathcal{L}(Y;Z)$, $U \subset X$ open, and $k \in \mathbb{N}$. Let $f\colon U \to Y$ be k-times Fréchet differentiable at $a \in U$. Then $D^k(L \circ f)(a) = L \circ D^k f(a)$ and consequently also $d^k(L \circ f)(a) = L \circ d^k f(a)$.*

We say that f is C^k-smooth if $D^k f$ (i.e. the mapping $x \mapsto D^k f(x)$) is continuous in the domain. Note that f is C^k-smooth if and only if $d^k f$ is continuous in the domain by Fact 1. We denote by $C^k(U;Y)$ the vector space of all C^k-smooth mappings from U into Y. Further, $C^0(U;Y) = C(U;Y)$ is the space of continuous mappings. We set $C^k(U) = C^k(U;\mathbb{R})$, $k \in \mathbb{N}_0$.

We also recall the following well-known corollary of the Taylor formula (it follows easily from [D, Theorem 8.14.2], or see [HJ, Corollary 1.108]).

Theorem 5 (see e.g. [D])**.** *Let X, Y be normed linear spaces, $U \subset X$ an open convex set, $k \in \mathbb{N}$, and $f \in C^k(U;Y)$. Then for any $x \in U$ and $h \in X$ satisfying $x+h \in U$ we have*

$$\left\| f(x+h) - \sum_{j=0}^{k} \frac{1}{j!} d^j f(x)[h] \right\| \le \frac{1}{k!} \Bigl(\sup_{t \in [0,1]} \bigl\| d^k f(x+th) - d^k f(x) \bigr\| \Bigr) \cdot \|h\|^k.$$

Let (P,ρ), (Q,σ) be metric spaces. The minimal modulus of continuity of a uniformly continuous mapping $f\colon P \to Q$ is defined as $\omega_f(\delta) = \sup\{\sigma(f(x),f(y));\ x,y \in P, \rho(x,y) \le \delta\}$ for $\delta \in [0,+\infty)$. Clearly, ω_f is continuous at 0.

A modulus is a non-decreasing function $\omega\colon [0,+\infty) \to [0,+\infty]$ continuous at 0 with $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$. (Notice that a modulus by our definition can be infinite on some interval.) We say that $f : P \to Q$ is uniformly continuous with modulus of continuity $\omega \in \mathcal{M}$ if $\omega_f \le \omega$.

Let X, Y be normed linear spaces, $U \subset X$ an open set, $k \in \mathbb{N}$, $f \in C^k(U;Y)$, and let $\omega \in \mathcal{M}$. We say that f is $C^{k,\omega}$-smooth on U (or $f \in C^{k,\omega}(U;Y)$) if $d^k f$ is uniformly continuous on U with modulus ω. Note that Fact 1 implies that if f is $C^{k,\omega}$-smooth, then $D^k f$ is uniformly continuous on U with modulus $\frac{k^k}{k!}\omega$, and conversely if f is C^k-smooth and $D^k f$ is uniformly continuous on U with modulus ω, then f is $C^{k,\omega}$-smooth.

Further, we define the kth modulus of smoothness of $f : U \to Y$, $U \subset X$ any set, by

$$\omega_k(f;t) = \sup_{\substack{\|h\|\le t\\ [x,x+kh]\subset U}} \|\Delta_h^k f(x)\|, \quad t \in [0,+\infty),$$

where $[x, x+kh]$ denotes the segment with endpoints x and $x+kh$. We remark that unlike in the definition of modulus of continuity, in the definition of modulus of smoothness we are not allowed to "jump over the gaps in the domain".

We will use the following version of Marchaud's theorem:

Theorem 6 ([M], [DL]). *Let $k \in \mathbb{N}$. There is a constant $A_k > 0$ such that if $a, b \in \mathbb{R}$, $a < b$, and $f : [a,b] \to \mathbb{R}$ is a bounded function, then f is k-times differentiable on (a,b) and for each $t \ge 0$*

$$\omega_{f^{(k)}}(t) = \omega_1(f^{(k)};t) \le A_k \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s,$$

provided that the integral on the right-hand side is finite for some $t > 0$.

For a continuous f it is proved e.g. in [DL, Theorem 6.3.1] (where we put $A = [a,b]$, $p = \infty$, and $r = k+1$); further, the original Marchaud's version [M, § 29, cf. § 22] shows that the finiteness of the integral already gives the continuity of f.

3. Converse Taylor theorems and simple consequences

Let X, Y be normed linear spaces, $U \subset X$ open, $f : U \to Y$, and $k \in \mathbb{N}_0$. We say that f is T^k-smooth at $x \in U$ if there exists a polynomial $P^x \in \mathcal{P}^k(X;Y)$ satisfying $P^x(0) = f(x)$ and

$$f(x+h) - P^x(h) = o(\|h\|^k),\ h \to 0.$$

We say that f is T^k-smooth on U if it is T^k-smooth at every point $x \in U$. Recall that by Peano's form of Taylor's theorem a C^k-smooth mapping is also T^k-smooth and the approximating polynomial is given by $\sum_{j=0}^k \frac{1}{j!} d^j f(x)$. A converse statement is contained in the following theorem, see e.g. [LS], [AD], cf. also [J] or [HJ, Theorem 1.110].

Theorem 7 (Converse Taylor theorem; [LS], [AD]). *Let X, Y be normed linear spaces, $U \subset X$ an open set, $f : U \to Y$, and $k \in \mathbb{N}_0$. Then $f \in C^k(U;Y)$ if and only if f is a T^k-smooth mapping satisfying*

$$\lim_{\substack{(y,h)\to(x,0)\\ h\ne 0}} \frac{\|R(y,h)\|}{\|h\|^k} = 0$$

for every $x \in U$, where $R(x,h) = f(x+h) - P^x(h)$ and the polynomials $P^x \in \mathcal{P}^k(X;Y)$ come from the definition of T^k-smoothness of f at x.

We will rely on a quantitative version of the above theorem. Let X, Y be normed linear spaces, $U \subset X$ an open set, $f : U \to Y$, and $k \in \mathbb{N}_0$. We say that f is UT^k-smooth on U with modulus ω if for each $x \in U$ there is a polynomial $P^x \in \mathcal{P}^k(X;Y)$ satisfying

$$\|f(x+h) - P^x(h)\| \le \omega(\|h\|)\|h\|^k \quad \text{for } x+h \in U.$$

For a convex bounded subset U of a normed linear space we define its "ellipticity" $e_U = \frac{\operatorname{diam} U}{\sup\{r;\ \exists a\in U:\ B(a,r)\subset U\}}$.

Theorem 8 ([J], see also [HJ, Theorem 1.125]). *Let X, Y be normed linear spaces, $U \subset X$ an open convex bounded set, $f : U \to Y$, and $k \in \mathbb{N}$. Suppose that f is UT^k-smooth on U with modulus ω. Then f is $C^{k,m\omega}$-smooth on U with $m = c_k e_U^k$, where $c_k > 0$ is a constant depending only on k.*

We note that the quantity e_U is defined slightly differently in [J], but it is clear that it is not larger than e_U as defined here.

Now we are ready to present our main tool. Let X, Y be normed linear spaces, $A \subset X$, $f : A \to Y$, and $k \in \mathbb{N}_0$. We say that f is weakly T^k-smooth at $x \in A$ if there exists a polynomial $P^x \in \mathcal{P}^k(X;Y)$ satisfying $P^x(0) = f(x)$ and $f(x+th) - P^x(th) = o(t^k)$, $t \to 0$ for each $h \in X$. We say that f is weakly T^k-smooth on A if it is weakly T^k-smooth at every $x \in A$.

We say that f is directionally T^k-smooth at $x \in A$ if the mapping $t \mapsto f(x+th)$ is T^k-smooth at 0 for each $h \in X$, i.e. for each $h \in X$ there is a polynomial $P^{x,h} \in \mathcal{P}^k(\mathbb{R};Y)$ satisfying $P^{x,h}(0) = f(x)$ and $f(x+th) - P^{x,h}(t) = o(t^k)$, $t \to 0$. We say that f is directionally T^k-smooth on A if it is directionally T^k-smooth at every $x \in A$.

Recall that all the polynomials in the above definitions are uniquely determined. Consequently, if f is directionally T^k-smooth at x, then $P^{x,h}(t) = P^{x,th}(1)$ for every $h \in X$ and $t \in \mathbb{R}$. Further, if f is weakly T^k-smooth at x, then f is directionally

T^k-smooth at x and for the approximating polynomials from the definitions the following holds: $P^{x,h}(t) = P^x(th)$ for $h \in X$, $t \in \mathbb{R}$. The converse implication holds under additional assumptions:

Theorem 9. *Let X be a Banach space, Y a normed linear space, $U \subset X$ open, $f : U \to Y$, $a \in U$, and $k \in \mathbb{N}$. Suppose that $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$, f is directionally T^k-smooth on U, and further*

$$R(a + ty, th) = o(t^k),\ t \to 0 \quad \text{for each } y, h \in X, \tag{3}$$

where $R(x, h) = f(x + h) - P^{x,h}(1)$ and where the polynomials $P^{x,h} \in \mathcal{P}^k(\mathbb{R}; Y)$ come from the definition of the directional T^k-smoothness of f at x. Then f is weakly T^k-smooth at a.

Proof. Without loss of generality we may assume that $a = 0$. Set $P(h) = P^{0,h}(1)$ for each $h \in X$. It is easy to see that it suffices to show that $P \in \mathcal{P}^k(X; Y)$, which will be done using Theorem 2. First we show that $\phi \circ P$ is Baire measurable for each $\phi \in Y^*$. So fix $\phi \in Y^*$. For each $h \in X$ there are $g_0(h), \dots, g_k(h) \in \mathbb{R}$ such that $\phi \circ P^{0,h}(t) = \sum_{j=0}^k g_j(h)t^j$ for $t \in \mathbb{R}$. Note that $\phi \circ P(h) = \phi \circ P^{0,h}(1) = \sum_{j=0}^k g_j(h)$ and so it suffices to show that the functions $g_0, \dots, g_k : X \to \mathbb{R}$ are Baire measurable. Clearly $g_0(h) = \phi \circ f(0)$ for each $h \in X$. We proceed by induction, assuming that $g_0, \dots, g_{m-1}$ are Baire measurable for some $1 \le m \le k$. Since for a fixed $h \in X$ we have

$$\lim_{t\to 0} \frac{1}{t^m}\left(\phi \circ f(th) - \sum_{j=0}^k g_j(h)t^j\right) = \phi\left(\lim_{t\to 0} \frac{f(th) - P^{0,h}(t)}{t^m}\right) = 0,$$

it follows that $g_m(h) = \lim_{t\to 0} \frac{1}{t^m}\bigl(\phi \circ f(th) - \sum_{j=0}^{m-1} g_j(h)t^j\bigr)$. In particular,

$$g_m(h) = \lim_{n\to\infty} n^m\left(\phi \circ f\left(\frac{h}{n}\right) - \sum_{j=0}^{m-1} \frac{g_j(h)}{n^j}\right).$$

Hence on each bounded set g_m is a pointwise limit of a sequence of Baire measurable functions and so it is Baire measurable.

Now we show that P satisfies the formula (2). Fix $x, h \in X$ and define

$$q(t) = \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P(tx + jth).$$

Clearly $q \in \mathcal{P}^k(\mathbb{R}; Y)$. Let $r > 0$ be such that $t(x + jh) \in U$ for all $j \in \{0, \dots, k+1\}$ and $|t| \le r$. We have

$$\begin{aligned}\|q(t)\| &= \left\| \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} \Bigl(P(tx + jth) - f(tx + jth) + f(tx + jth) - P^{tx,th}(j) + P^{tx,th}(j)\Bigr) \right\| \\ &\le \sum_{j=0}^{k+1} \binom{k+1}{j} \bigl(\|R(0, tx + jth)\| + \|R(tx, jth)\|\bigr)\end{aligned}$$

for all $|t| \le r$, since $\sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} P^{tx,th}(j) = 0$ by Theorem 2. This estimate combined with the assumption (3) implies that $q(t) = o(t^k),\ t \to 0$. It follows that q is a zero polynomial (apply the classical fact to $\phi \circ q$, $\phi \in Y^*$), and in particular $q(1) = 0$, which is what we wanted to prove.

□

As a consequence of the above theorem we obtain that the Converse Taylor theorem holds even if we assume only directional T^k-smoothness:

Theorem 10. *Let X be a Banach space, Y a normed linear space, $U \subset X$ an open set, $f : U \to Y$, and $k \in \mathbb{N}$. Then $f \in C^k(U; Y)$ if and only if f is a directionally T^k-smooth mapping such that $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$ and*

$$\lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|R(y, h)\|}{\|h\|^k} = 0 \tag{4}$$

for every $x \in U$, where $R(x, h) = f(x + h) - P^{x,h}(1)$ and the polynomials $P^{x,h} \in \mathcal{P}^k(X; Y)$ come from the definition of the directional T^k-smoothness of f at x.

Proof. It suffices to notice that the assumption (4) implies (3) in Theorem 9 and it also promotes the weak T^k-smoothness to T^k-smoothness. Hence both implications follow from Theorem 7.

□

We end this section by showing that Theorem 9 and Theorem 8 (and the one-dimensional Marchaud's theorem) very easily imply Marchaud's theorem in higher (even infinite) dimensions under the assumption of continuity of f. In the next section we give more general versions of the propositions below, which need some additional technical results.

We believe that the following simple proof of Proposition 13 may be interesting also for people working only with functions on $\mathbb{R}^n$. We underline that Proposition 13 can be very easily generalised (see Remark 14) to more general subsets of $\mathbb{R}^n$ than the bounded convex sets (namely the sets with the UCC property); in this way we obtain a result that is more general (see Proposition 21) than the currently known multi-dimensional versions of Marchaud's theorem ([JS], which works with LG domains).

Definition 11. Let X, Y be normed linear spaces and $U \subset X$ an open set. We say that $f : U \to Y$ is $C^{k,\omega}$-smooth on every open segment in U if for every $x \in U$, $h \in S_X$, and $(a,b) \subset \mathbb{R}$ satisfying $x + th \in U$ for all $t \in (a,b)$ the mapping $g\colon (a,b) \to Y$, $g(t) = f(x+th)$, is $C^{k,\omega}$-smooth.

Proposition 12. *Let X be a Banach space, Y a normed linear space, $U \subset X$ an open convex bounded set, $f \in C(U;Y)$, $k \in \mathbb{N}$, and $\omega \in \mathcal{M}$. Suppose that f is $C^{k,\omega}$-smooth on every open segment in U. Then f is $C^{k,m\omega}$-smooth on U with $m = c_k e_U^k$, where $c_k > 0$ is a constant depending only on k.*

Proof. Clearly f is directionally T^k-smooth on U. Let $x, x+h \in U$, $h \neq 0$. Then $g(t) = f\big(x + t\frac{h}{\|h\|}\big)$ is $C^{k,\omega}$-smooth on $(-\delta, \|h\|+\delta)$ for some $\delta > 0$. Thus Taylor's theorem (Theorem 5) used on g at 0 with the increment $\|h\|$ implies that $\|R(x,h)\| = \|f(x+h) - P^{x,h}(1)\| = \big\|g(\|h\|) - P^{x,h/\|h\|}(\|h\|)\big\| \le \frac{1}{k!}\omega(\|h\|)\|h\|^k$, where the polynomials $P^{x,h} \in \mathcal{P}^k(\mathbb{R};Y)$ come from the definition of the directional T^k-smoothness. So (3) is clearly satisfied at each $a \in U$ and thus f is weakly T^k-smooth on U by Theorem 9. Consequently, f is UT^k-smooth on U with modulus $\frac{1}{k!}\omega$, so the application of Theorem 8 finishes the proof.

□

Proposition 13. *Let X be a Banach space, $U \subset X$ an open convex bounded set, $f \in C(U)$, and $k \in \mathbb{N}$. Then $f \in C^k(U)$ and*

$$\omega_{d^k f}(t) \le B_k e_U^k \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s,$$

provided that the integral on the right-hand side is finite for some $t > 0$. The constant B_k depends only on k.

Proof. For a fixed $x \in U$ and $h \in S_X$ we define $g(t) = f(x+th)$ for $t \in (c,d)$, where (c,d) is the maximal interval such that $x+th \in U$ for all $t \in (c,d)$. Then it is easy to see (using (1)) that $\omega_{k+1}(g;s) \le \omega_{k+1}(f;s)$ for all $s \in [0,+\infty)$. Therefore $\omega_{g^{(k)}}(t) \le A_k \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s$ by Theorem 6 used on each $[a,b] \subset (c,d)$. Hence Proposition 12 implies that $\omega_{d^k f}(t) \le A_k c_k e_U^k \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s$.

□

Remark 14. If U has the UCC property (see Definition 18), then estimating as in (7) we obtain the assertion of the above proposition with

$$\omega_{d^k f}(t) \le C_{k,U} \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\,\mathrm{d}s,$$

where the constant $C_{k,U}$ depends only on k and U.

4. Main results

Remark 15. First we remark that Theorem 9 holds also for incomplete spaces X provided that we additionally assume that $\phi \circ f$ is continuous for each $\phi \in Y^*$ and

$$\lim_{\substack{(y,t)\to(h,0)\\ t\neq 0}} \frac{\|R(a,ty)\|}{t^k} = 0 \quad \text{for each } h \in X. \tag{5}$$

Indeed, it suffices to modify the proof of Theorem 9 by showing that in this case $\phi \circ P$ is continuous for each $\phi \in Y^*$ and then use the remark after Theorem 2. It is clear by passing to $\phi \circ f$ and $\phi \circ P$ that it suffices to prove that P is continuous if f is continuous in case that $Y = \mathbb{R}$. So, suppose to the contrary that there are $\varepsilon > 0$ and $\{h_n\} \subset X$ such that $h_n \to h \in X$ and $|P(h_n) - P(h)| \ge \varepsilon$. Put $q_n(t) = P(th_n) - P(th)$, $n \in \mathbb{N}$. Then $q_n(t) = P^{0,h_n}(t) - P^{0,h}(t)$ and so $q_n \in \mathcal{P}^k(\mathbb{R};\mathbb{R})$; recall that $a = 0$. Let $K = \bigcup_{|t|\le r} t\big(\{h\} \cup \{h_n;\ n \in \mathbb{N}\}\big)$ for some $r > 0$ such that $K \subset U$. Then K is compact and hence f is uniformly continuous on K. Let ω be a modulus of continuity of f on K. We have

$$|q_n(t)| \le |P(th_n) - f(th_n)| + |f(th_n) - f(th)| + |f(th) - P(th)| \le |R(0,th_n)| + \omega(\|th_n - th\|) + |R(0,th)|$$

for $|t| \le r$. Recall the following elementary fact about polynomials: For a given $k \in \mathbb{N}$ there is $M > 0$ such that if $Q \in \mathcal{P}^k(\mathbb{R};\mathbb{R})$, then $|Q(x)| \le M\|Q\||x|^k$ for every $x \in \mathbb{R}$, $|x| \ge 1$ (it follows easily e.g. from [HJ, Fact 1.42]). By (5) there are $0 < \delta \le \min\{r, 1\}$ and $n_0 \in \mathbb{N}$ such that $|R(0,th_n)| \le \frac{\varepsilon}{4M}|t|^k$ and $|R(0,th)| \le \frac{\varepsilon}{4M}|t|^k$ for all $|t| \le \delta$ and $n \ge n_0$. Hence, applying the above fact to $Q(t) = q_n(\delta t)$, we obtain

$$\varepsilon \le |q_n(1)| \le \frac{M}{\delta^k}\sup_{|t|\le\delta}|q_n(t)| \le \frac{M}{\delta^k}\Big(\sup_{|t|\le\delta}|R(0,th_n)| + \omega(\delta\|h_n - h\|) + \sup_{|t|\le\delta}|R(0,th)|\Big) \le \frac{\varepsilon}{2} + \frac{M}{\delta^k}\omega(\delta\|h_n - h\|)$$

for all $n \ge n_0$. This is a contradiction.

We will need the following easy fact, which is well-known for $X = Y = \mathbb{R}$ (see e.g. [DS, p. 215, (5.2)]).

Proposition 16. *For each $k \in \mathbb{N}$, $\omega \in \mathcal{M}$, $C > 0$, $R \geq r > 0$ such that $\omega(R) < +\infty$, and $j \in \{1, \dots, k\}$ there is $M_j(k, \omega, r, R, C) > 0$ such that if X, Y are normed linear spaces, $U \subset X$ is an open convex set such that $U(a, r) \subset U \subset U(a, R)$ for some $a \in U$, and $f \in C^{k,\omega}(U; Y)$ satisfies $\|f(x)\| \leq C$ for $x \in U(a, r)$, then $\|d^j f(x)\| \leq M_j(k, \omega, r, R, C)$ for $x \in U$.*

Proof. First we claim that for $j = k$ we can take $M_k(k, \omega, r, R, C) = \frac{(4k)^k C}{r^k} + \omega(r) + \omega(R)$. For a fixed $h \in S_X$ and $\phi \in B_{Y^*}$ we put $g(t) = \phi \circ f(a + th)$ for $t \in (-r, r)$. Using Facts 3 and 4 it is easy to see that $g \in C^{k,\omega}\big((-r, r)\big)$. Put $s = \frac{r}{2k}$. It is a well-known fact (see e.g. [Z, Exercise 5.3.14] or [DS, p. 195, (3.34)]) that there is $\xi \in (0, r)$ such that $g^{(k)}(\xi) = \frac{\Delta_s^k g(0)}{s^k}$. It follows from (1) that $|g^{(k)}(\xi)| \leq \frac{2^k C}{s^k}$ and so $|g^{(k)}(0)| \leq |g^{(k)}(\xi)| + |g^{(k)}(0) - g^{(k)}(\xi)| \leq \frac{2^k C}{s^k} + \omega(r)$. Using Facts 3 and 4 again we get $\big|\phi\big(d^k f(a)[h]\big)\big| = |g^{(k)}(0)| \leq \frac{(4k)^k C}{r^k} + \omega(r)$. By taking the supremum over all $h \in S_X$ and $\phi \in B_{Y^*}$ we finally obtain $\|d^k f(a)\| \leq \frac{(4k)^k C}{r^k} + \omega(r)$. Now for any $x \in U$ we have $\|d^k f(x)\| \leq \|d^k f(a)\| + \|d^k f(x) - d^k f(a)\| \leq M_k(k, \omega, r, R, C)$.

Next, fix $j \in \mathbb{N}$, $C > 0$, and $R \geq r > 0$. It is sufficient to show the existence of $M_j(k, \omega, r, R, C)$ for $k \geq j$ and $\omega \in \mathcal{M}$ with $\omega(R) < +\infty$. We have already proved it for $k = j$. Now assume that $k > j$ and $M_j(k-1, \tilde{\omega}, r, R, C)$ exists for all $\tilde{\omega} \in \mathcal{M}$ with $\tilde{\omega}(R) < +\infty$. To prove the existence of $M_j(k, \omega, r, R, C)$ let $\omega \in \mathcal{M}$ and $f \in C^{k,\omega}(U; Y)$ satisfying the assumptions be given. We already know that $\|d^k f(x)\| \leq M_k(k, \omega, r, R, C)$ for $x \in U$. Hence by Fact 1, $\|D^k f(x)\| \leq \frac{k^k}{k!} M_k(k, \omega, r, R, C)$ for $x \in U$. It follows that $D^{k-1} f$ (and consequently also $d^{k-1} f$) is Lipschitz with constant $\frac{k^k}{k!} M_k(k, \omega, r, R, C)$ and so $f \in C^{k-1,\tilde{\omega}}(U; Y)$, where $\tilde{\omega}(t) = \frac{k^k}{k!} M_k(k, \omega, r, R, C) \cdot t$. Thus we may set $M_j(k, \omega, r, R, C) = M_j(k-1, \tilde{\omega}, r, R, C)$. □

Let X be a normed linear space. Recall that a set $A \subset B_{X^*}$ is called λ-norming, $\lambda \geq 1$, if $\sup_{\phi \in A} \phi(x) \geq \frac{1}{\lambda}\|x\|$ for each $x \in X$.

Theorem 17. *Let X, Y be normed linear spaces, $U \subset X$ open, $f : U \to Y$ a locally bounded mapping, $k \in \mathbb{N}$, $\omega \in \mathcal{M}$, and $A \subset B_{Y^*}$ a λ-norming set. If $\phi \circ f \in C^{k,\omega}(U)$ for each $\phi \in A$, then $f \in C^{k,\lambda\omega}(U; Y)$.*

Proof. By passing to the completion of Y we may assume without loss of generality that Y is a Banach space.

We prove the theorem by induction on k. More precisely, let X be a normed linear space and $U \subset X$ an open set; for each $k \in \mathbb{N}$ we prove the following statement: If Y is a Banach space, $A \subset B_{Y^*}$ a λ-norming set, $\omega \in \mathcal{M}$, and $f : U \to Y$ a locally bounded mapping satisfying $\phi \circ f \in C^{k,\omega}(U)$ for each $\phi \in A$, then $f \in C^{k,\lambda\omega}(U; Y)$.

First assume that $k = 1$. Fix $x \in U$. To prove that the derivative $Df(x)$ exists we first show that for a given $h \in X$ the limit $L(h) = \lim_{t \to 0} \frac{1}{t}\big(f(x + th) - f(x)\big)$ exists using the Cauchy criterion. By the Mean value theorem, for any $\phi \in A$, $\delta > 0$ sufficiently small, and $s, t \in (-\delta, \delta) \setminus \{0\}$ there are $\theta, \xi \in (-\delta, \delta)$ such that

$$\begin{aligned}\left|\frac{1}{t}\big(\phi \circ f(x + th) - \phi \circ f(x)\big) - \frac{1}{s}\big(\phi \circ f(x + sh) - \phi \circ f(x)\big)\right| &= \big|D(\phi \circ f)(x + \theta h)[h] - D(\phi \circ f)(x + \xi h)[h]\big| \\ &\leq \|h\|\omega(2\|h\|\delta).\end{aligned}$$

Hence for $s, t \in (-\delta, \delta) \setminus \{0\}$

$$\begin{aligned}\left\|\frac{1}{t}\big(f(x + th) - f(x)\big) - \frac{1}{s}\big(f(x + sh) - f(x)\big)\right\| &\leq \lambda \sup_{\phi \in A}\left|\phi\left(\frac{1}{t}\big(f(x + th) - f(x)\big) - \frac{1}{s}\big(f(x + sh) - f(x)\big)\right)\right| \\ &= \lambda \sup_{\phi \in A}\left|\frac{1}{t}\big(\phi \circ f(x + th) - \phi \circ f(x)\big) - \frac{1}{s}\big(\phi \circ f(x + sh) - \phi \circ f(x)\big)\right| \leq \lambda\|h\|\omega(2\|h\|\delta).\end{aligned} \tag{6}$$

This clearly implies the existence of $L(h)$. From the definition of $L(h)$ clearly $\phi(L(h)) = D(\phi \circ f)(x)[h]$ for any $\phi \in A$. Since A separates the points of Y, it easily follows that L is linear. Finally, for any $h \in X$ sufficiently small we can set $t = 1$ and pass to the limit as $s \to 0$ in (6) to obtain $\|f(x+h) - f(x) - L(h)\| \leq \lambda\omega(2\|h\|)\|h\|$, which implies that L is bounded and $Df(x) = L$. Also, for any $x, y \in U$ using Fact 4 we can estimate

$$\big\|Df(x) - Df(y)\big\| \leq \sup_{h \in B_X} \lambda \sup_{\phi \in A}\big|\phi\big(Df(x)[h] - Df(y)[h]\big)\big| = \lambda \sup_{\phi \in A}\sup_{h \in B_X}\big|D(\phi \circ f)(x)[h] - D(\phi \circ f)(y)[h]\big| \leq \lambda\omega(\|x - y\|).$$

To continue the induction, assume that the statement holds for $k - 1$ and the assumptions of the statement for k are satisfied. Fix $x \in U$ and let $r, C > 0$ be such that $\|f(y)\| \leq C$ for $y \in U(x, r)$ and $\omega(r) < +\infty$. By Proposition 16 for each $\phi \in A$ the mapping $d^k(\phi \circ f)$ is bounded by $M_k(k, \omega, r, r, C)$ on $U(x, r)$. Hence $\|D^k(\phi \circ f)(y)\| \leq \frac{k^k}{k!} M_k(k, \omega, r, r, C)$ for $y \in U(x, r)$ by Fact 1. It follows that $D^{k-1}(\phi \circ f)$, and consequently also $d^{k-1}(\phi \circ f)$, is Lipschitz with constant $\frac{k^k}{k!} M_k(k, \omega, r, r, C)$ on $U(x, r)$. Thus the inductive hypothesis (used for the modulus $\tilde{\omega}(t) = \frac{k^k}{k!} M_k(k, \omega, r, r, C) \cdot t$) yields that $f \in C^{k-1}\big(U(x, r); Y\big)$. Let $W \subset \big(\mathcal{L}({}^{k-1}X; Y)\big)^*$ be the set of all functionals for which there exist $\phi \in A$ and $h_1, \dots, h_{k-1} \in B_X$ such that $\psi(M) = \phi\big(M(h_1, \dots, h_{k-1})\big)$ for $M \in \mathcal{L}({}^{k-1}X; Y)$. Then W is clearly a λ-norming set. We set $g = D^{k-1} f$ on $U(x, r)$. For any $\psi \in W$ determined by $\phi \in A$ and $h_1, \dots, h_{k-1} \in B_X$, and any $y \in U(x, r)$ using Fact 4 we obtain $(\psi \circ g)(y) = \big(\phi \circ D^{k-1} f(y)\big)[h_1, \dots, h_{k-1}] = \big(D^{k-1}(\phi \circ f)(y)\big)[h_1, \dots, h_{k-1}] = \big(\varepsilon \circ D^{k-1}(\phi \circ f)\big)(y)$, where $\varepsilon \in \big(\mathcal{L}({}^{k-1}X; \mathbb{R})\big)^*$ is the operator of evaluation $\varepsilon(M) = M(h_1, \dots, h_{k-1})$. Thus $D(\psi \circ g)(y)[h] = D\big(\varepsilon \circ D^{k-1}(\phi \circ f)\big)(y)[h] = \varepsilon\big(D(D^{k-1}(\phi \circ f))(y)[h]\big) = D^k(\phi \circ f)(y)[h, h_1, \dots, h_{k-1}]$, again by Fact 4. From Fact 1 it follows that $\psi \circ g \in C^{1,m\omega}(U(x, r))$, where $m = \frac{k^k}{k!}$. Thus by the first step of the induction applied to the (continuous) mapping g, the λ-norming set W, and the modulus $\bar{\omega} = m\omega$ we obtain that $Dg = D^k f$ exists on $U(x, r)$.

To finish the proof, for any $x, y \in U$ using Fact 4 we can estimate

$$\left\|d^k f(x) - d^k f(y)\right\| \le \sup_{h \in B_X} \lambda \sup_{\phi \in A} \left|\phi\left(d^k f(x)[h] - d^k f(y)[h]\right)\right| = \lambda \sup_{\phi \in A} \sup_{h \in B_X} \left|d^k(\phi \circ f)(x)[h] - d^k(\phi \circ f)(y)[h]\right| \le \lambda\omega(\|x - y\|).$$

□

Definition 18. Let $N \in \mathbb{N}$ and $e > 0$. We say that an open subset U of a normed linear space has the (N, e)-uniform convex chain property (or (N, e)-UCC property) if for each $x, y \in U$ there is a polygonal path $[x_0, \dots, x_n]$, $n \le N$, with $x = x_0$ and $y = x_n$ such that $\|x_j - x_{j-1}\| \le \|x - y\|$ and the segment $[x_{j-1}, x_j]$ lies in an open convex bounded $V_j \subset U$ with $e_{V_j} \le e$ for each $j = 1, \dots, n$. We say that U has the uniform convex chain property (or UCC property) if it has the (N, e)-UCC property for some $N \in \mathbb{N}$ and $e > 0$.

Theorem 19. *Let X, Y be normed linear spaces, $U \subset X$ an open set with the UCC property, $f : U \to Y$, $k \in \mathbb{N}$, and $\omega \in \mathcal{M}$. Suppose that f is $C^{k,\omega}$-smooth on every open segment in U and that either*

(i) f is locally bounded, or
(ii) X is complete and $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$.

Then f is $C^{k,m\omega}$-smooth on U for some $m > 0$. More precisely, if U has the (N, e)-UCC property, then we can set $m = c_k N e^k$, where $c_k > 0$ is a constant depending only on k.

Proof. First assume that U is additionally convex and bounded. Clearly f is directionally T^k-smooth on U. Let $x, x + h \in U$, $h \ne 0$. Then $g(t) = f\left(x + t\frac{h}{\|h\|}\right)$ is $C^{k,\omega}$-smooth on $(-\delta, \|h\| + \delta)$ for some $\delta > 0$. Thus Taylor's theorem (Theorem 5) used on g at 0 with the increment $\|h\|$ implies that $\|R(x, h)\| = \|f(x + h) - P^{x,h}(1)\| = \left\|g(\|h\|) - P^{x,h/\|h\|}(\|h\|)\right\| \le \frac{1}{k!}\omega(\|h\|)\|h\|^k$, where the polynomials $P^{x,h} \in \mathcal{P}^k(\mathbb{R}; Y)$ come from the definition of the directional T^k-smoothness. If (ii) holds, then since (3) is clearly satisfied at each $a \in U$, the mapping f is weakly T^k-smooth on U by Theorem 9.

If (i) holds, then f is continuous. Indeed, for a fixed $a \in U$ we find $r > 0$ such that $U(a, r) \subset U$, f is bounded by $C > 0$ on $U(a, r)$, and $\omega(r) < +\infty$. Then Proposition 16 implies that for each $v \in S_X$ the $C^{k,\omega}$-smooth mapping $t \mapsto f(a + tv)$ is Lipschitz on $(-r, r)$ with constant $M_1(k, \omega, r, r, C)$. Further, since (3) and (5) are clearly satisfied at each $a \in U$, Remark 15 implies that f is weakly T^k-smooth on U. Consequently, in both cases f is UT^k-smooth on U with modulus $\frac{1}{k!}\omega$, so an application of Theorem 8 yields that f if $C^{k,m\omega}$ smooth with $m = c_k e_U^k$, where c_k depends only on k.

Now assume that U has the (N, e)-UCC property. Fix any $x, y \in U$ and let $[x_0, \dots, x_n]$, $n \le N$, be the polygonal path and $V_1, \dots, V_n$ the convex sets from the definition of the UCC property. Then using the first part of the proof on each of the sets V_j we obtain

$$\|d^k f(x) - d^k f(y)\| \le \sum_{j=1}^{n} \|d^k f(x_j) - d^k f(x_{j-1})\| \le \sum_{j=1}^{n} c_k e_{V_j}^k \omega(\|x_j - x_{j-1}\|) \le N c_k e^k \omega(\|x - y\|). \tag{7}$$

□

Combining the one-dimensional Marchaud's theorem together with our criterions for $C^{k,\omega}$-smoothness we obtain the following general version of Marchaud's theorem:

Theorem 20. *Let X, Y be normed linear spaces, $U \subset X$ an open set with the (N, e)-UCC property, $f : U \to Y$, and $k \in \mathbb{N}$. Suppose that either*

(i) f is locally bounded, or
(ii) X is complete, f is bounded on every closed segment in U, and $\phi \circ f$ is Baire measurable for each $\phi \in Y^*$.

Then $f \in C^k(U; Y)$ and for each $t \ge 0$

$$\omega_{d^k f}(t) \le B_k N e^k \int_0^t \frac{\omega_{k+1}(f; s)}{s^{k+1}}\, \mathrm{d}s,$$

provided that the integral on the right-hand side is finite for some $t > 0$. The constant B_k depends only on k.

Proof. For a fixed $x \in U$ and $h \in S_X$ we define $g(t) = f(x + th)$ for $t \in (c, d)$, where (c, d) is the maximal interval containing 0 such that $x + th \in U$ for all $t \in (c, d)$. Further, we set $g_\phi = \phi \circ g$ for each $\phi \in B_{Y^*}$. Then it is easy to see (using (1)) that $\omega_{k+1}(g_\phi; s) \le \omega_{k+1}(g; s) \le \omega_{k+1}(f; s)$ for all $s \in [0, +\infty)$. Thus $\omega_{g_\phi^{(k)}}(t) \le \eta(t) = A_k \int_0^t \frac{\omega_{k+1}(f;s)}{s^{k+1}}\, \mathrm{d}s$ by Theorem 6 used on each $[a, b] \subset (c, d)$. It follows that $g \in C^{k,\eta}\big((c, d)\big)$ by Theorem 17. Hence Theorem 19 implies that $\omega_{d^k f}(t) \le c_k N e^k \eta(t)$.

□

To put the UCC property into perspective, we recall that the strongest currently known formulation of Marchaud's theorem is for the bounded LG domains in $\mathbb{R}^n$, which follows from [JS, Theorems 1 and 2]; notice that $0 \le t \le 1$ in [JS, Theorem 1], so the case of unbounded domains is not covered. It is easily seen that each LG domain as defined in [JS, p. 123] satisfies the uniform cone condition of [AF, p. 83]. For our purposes it is sufficient to deal only with bounded domains, in which case the definition of the uniform cone condition simplifies to the following: An open bounded subset U of $\mathbb{R}^n$ is said to satisfy the uniform cone condition if there exist $\delta > 0$, a finite open covering $\{U_j\}_{j=1}^m$ of the set $\{x \in \overline{U};\ \operatorname{dist}(x, \mathbb{R}^n \setminus U) \le \delta\}$, and a corresponding sequence $\{C_j\}_{j=1}^m$ of cones all linearly isometric to a fixed cone with vertex at the origin, such that $x + C_j \subset U$ for each $x \in U_j \cap U$.

Recall that a cone in $\mathbb{R}^n$ with vertex at the origin, axis direction $v \in \mathbb{R}^n$, $\|v\| = 1$, aperture angle $\varphi \in (0, \pi]$, and height $\rho > 0$ is the set $C = C(v, \varphi, \rho) = \{x \in \mathbb{R}^n;\ \langle x, v\rangle \ge \|x\| \cos\frac{\varphi}{2}, \|x\| \le \rho\}$.

The following proposition shows that in $\mathbb{R}^n$ our version of Marchaud's theorem is more general than that of [JS].

Proposition 21. *Let $U \subset \mathbb{R}^n$ be a connected open bounded set that satisfies the uniform cone condition. Then U has the UCC property. On the other hand, there is a bounded open set in $\mathbb{R}^2$ with the UCC property that does not satisfy the uniform cone condition (and even neither the segment condition nor the weak cone condition, see* [AF, pp. 82–84]*).*

Before proving the proposition we make a couple of simple observations:

(a) If U is an open connected subset of a normed linear space X and $K \subset U$ is compact, then there is an open connected set $V \subset X$ such that $K \subset V \subset U$ and $\operatorname{dist}(V, X \setminus U) > 0$. Indeed, K can be covered by finitely many balls $U(x_j, r_j)$, $j = 1, \dots, n$, such that $U(x_j, 2r_j) \subset U$. Since U is pathwise connected, there are curves $\gamma_j \subset U$ connecting x_j and x_1, $j = 1, \dots, n$. By the compactness of γ_j there are open connected neighbourhoods V_j of γ_j such that $\operatorname{dist}(V_j, X \setminus U) > 0$. Thus it suffices to put $V = \bigcup_{j=1}^n \bigl(U(x_j, r_j) \cup V_j\bigr)$.

(b) Let $C \subset \mathbb{R}^n$ be the cone $C = C(v, 2\varphi, \rho)$. If $x, y \in \mathbb{R}^n$ satisfy $0 < \|x - y\| < \rho\frac{\sin\varphi}{2+\sin\varphi}$, then $z = x + \frac{2}{\sin\varphi}\|x - y\|v \in (x + \operatorname{Int} C) \cap (y + \operatorname{Int} C)$. Indeed, clearly $z \in x + \operatorname{Int} C$. Further, $\|z - y\| \le \|z - x\| + \|x - y\| = \|x - y\|\bigl(\frac{2}{\sin\varphi} + 1\bigr) < \rho$. Finally, to estimate the angle γ between v and $z - y$ consider the triangle xyz. Then $\sin\gamma = \frac{h}{\|z-x\|}$, where h is the altitude of the triangle xyz from the vertex x, and so $\sin\gamma \le \frac{\|x-y\|}{\|z-x\|} = \frac{\sin\varphi}{2} < \sin\varphi$.

Proof of Proposition 21. Let $U \subset \mathbb{R}^n$ be a connected open bounded set satisfying the uniform cone condition. Then there is a cone $C \subset \mathbb{R}^n$ with vertex at the origin and $\delta > 0$ such that the set $K = \{x \in \overline{U};\ \operatorname{dist}(x, \mathbb{R}^n \setminus U) \le \delta\}$ is covered by finitely many open sets U_j, $j = 1, \dots, m$, such that $x + C_j \subset U$ for every $x \in U \cap U_j$, where C_j is a cone linearly isometric to C. Let σ be a Lebesgue number of the covering $\{U_j \cap K\}_{j=1}^m$ of K (see [E, Theorem 4.3.31]). Then $x + C_j \subset U$ and $y + C_j \subset U$ for some $j \in \{1, \dots, m\}$ whenever $x, y \in K \cap U$ are such that $\|x - y\| < \sigma$.

Let $C = C(u, 2\varphi, \rho)$ and let $0 < \eta \le \delta$ be such that C contains a closed ball of radius η. Put $W = \{x \in U;\ \operatorname{dist}(x, \mathbb{R}^n \setminus U) > \eta\}$. By (a) above there is an open connected $V \subset \mathbb{R}^n$ such that $W \subset V \subset U$ and $\zeta = \operatorname{dist}(V, \mathbb{R}^n \setminus U) > 0$, so $\zeta \le \eta \le \delta$. Let $d = \operatorname{diam} C$ and put $\varepsilon = \min\bigl\{\zeta, \sigma, \rho\frac{\sin\varphi}{2+\sin\varphi}\bigr\}$. By the compactness the set $\overline{V}$ is covered by balls $U(x_j, \frac{\varepsilon}{2})$, $x_j \in \overline{V}$, $j = 1, \dots, M$. We claim that U has the (N, e_C)-UCC property, where $N = \max\bigl\{\lceil\frac{2}{\sin\varphi}\rceil + \lceil\frac{2}{\sin\varphi} + 1\rceil, 2\lceil\frac{d}{\varepsilon}\rceil + M + 1\bigr\}$.

Let $x, y \in U$. We distinguish three cases. First assume that $\|x - y\| < \varepsilon$ and $x \in V$ or $y \in V$. If $x \in V$, then $y \in U(x, \zeta) \subset U$ and note that for any ball B we have $e_B = 2 < e_C$. Clearly, the polygonal path consisting of the segment $[x, y]$ lies in $U(x, \zeta)$. If $y \in V$, then we proceed analogously.

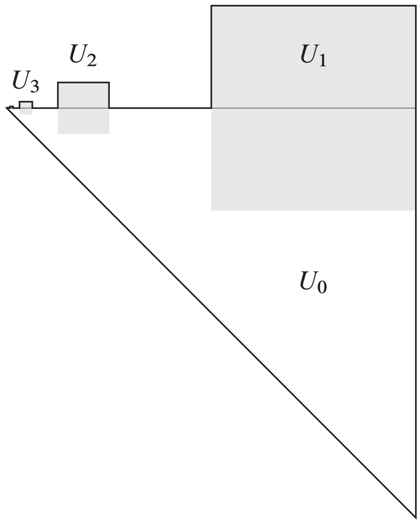
Next, if $\|x - y\| < \varepsilon$ and $x, y \in U \setminus V$, then $x, y \in K$ and since $\varepsilon \le \sigma$, there is a cone C_j such that $x + C_j \subset U$ and $y + C_j \subset U$. Let the axis of C_j be given by the vector $v \in \mathbb{R}^n$, $\|v\| = 1$. Set $z = x + \frac{2}{\sin\varphi}\|x - y\|v$. Then $z \in (x + \operatorname{Int} C_j) \cap (y + \operatorname{Int} C_j)$ by (b) above. Using the compactness of C_j we find $t < 0$ such that if we set $V_x = x + tv + \operatorname{Int} C_j$ and $V_y = y + tv + \operatorname{Int} C_j$, then $x, z \in V_x \subset U$ and $y, z \in V_y \subset U$. Thus by partitioning the segments $[x, z]$ and $[z, y]$ we can create a polygonal path between x and y consisting of $\lceil\frac{2}{\sin\varphi}\rceil + \lceil\frac{2}{\sin\varphi} + 1\rceil$ segments of length at most $\|x - y\|$ such that each of the segments lies in V_x or V_y. Clearly $e_{V_x} = e_{V_y} = e_C$.

Finally, we deal with the case $\|x - y\| \ge \varepsilon$. If $x \in V$, then we set $w_x = x$. Otherwise there is a cone C_j such that $x + C_j \subset U$. By the assumption there is $w_x \in U$ satisfying $B(w_x, \eta) \subset x + C_j$. It follows that $w_x \in W \subset V$. By shifting the cone slightly we obtain an open cone $V_x \subset U$ affinely isometric to $\operatorname{Int} C$ such that $x, w_x \in V_x$. Thus there is a polygonal path between x and w_x that lies in V_x and consists of at most $\bigl\lceil\frac{\|w_x - x\|}{\|x - y\|}\bigr\rceil \le \lceil\frac{d}{\varepsilon}\rceil$ segments of length at most $\|x - y\|$. Similarly we construct $w_y \in V$ and a polygonal path between y and w_y with analogous properties.

Now it suffices to notice that there is a polygonal path between w_x and w_y that consists of at most $M + 1$ segments of length at most $\varepsilon \le \|x - y\|$ such that each segment lies in a ball contained in U. Indeed, recall that $\overline{V}$ is covered by balls $U(x_j, \frac{\varepsilon}{2})$, $j = 1, \dots, M$, and each $U(x_j, \varepsilon) \subset U$. There are $p, q \in \mathbb{N}$ such that $w_x \in U(x_p, \frac{\varepsilon}{2})$ and $w_y \in U(x_q, \frac{\varepsilon}{2})$. Next, define $A_0 = U(x_p, \frac{\varepsilon}{2})$ and $A_l = \bigcup\bigl\{U(x_j, \frac{\varepsilon}{2});\ U(x_j, \frac{\varepsilon}{2}) \cap A_{l-1} \ne \emptyset\bigr\}$, $l \in \mathbb{N}$. Clearly, $A_l = A_{M-1}$ for $l \ge M$. Since V is connected, $x_q \in A_{M-1}$, and hence there is a polygonal path $[x_p, x_{k_1}, \dots, x_{k_s}, x_q]$, where $s \le M - 2$ and each of its segments is of length less than ε.

On the other hand, consider the following set: Let $U_0 \subset \mathbb{R}^2$ be the open triangle with vertices $[0, 0]$, $[1, 0]$, and $[1, -1]$, let $U_j = (2^{1-2j}, 2^{2-2j}) \times (-2^{-2j}, 2^{-2j})$, $j \in \mathbb{N}$, and $U = \bigcup_{j=0}^\infty U_j$. Then it is easy to see that for each $a_j = [2^{2-2j}, 2^{-2j}]$ there is no segment longer than $5 \cdot 2^{-2j}$ that lies in $\overline{U}$ and has a_j as an endpoint. Thus the set U does not satisfy the uniform cone condition (consider the open set from the open covering in the definition that contains the point $[0, 0]$ on the boundary of U). Similarly it can be seen that U neither satisfies the segment condition nor the weak cone condition of [AF]. On the other hand, U has the $\bigl(3, \frac{2}{\sqrt{2}-1}\bigr)$-UCC property. Indeed, $e_{U_0} = \frac{2}{\sqrt{2}-1}$ and $e_{U_j} = 2\sqrt{2} < e_{U_0}$, $j \in \mathbb{N}$. Now assume that $x = [x_1, x_2] \in U_j$, $y = [y_1, y_2] \in U_k$, $1 \le j < k$, and $x_2, y_2 \ge 0$. Put $u = [x_1, -a]$ and $v = [y_1, -a]$, where $a = \min\{2^{-2k-1}, 2^{-2j} - x_2\}$. Then $\|x - y\| \ge x_1 - y_1 > 2^{-2j}$ and so $\|u - x\| = x_2 + a \le 2^{-2j} < \|x - y\|$, $\|v - u\| = x_1 - y_1 \le \|x - y\|$, and $\|y - v\| = y_2 + a < 2^{-2k} + 2^{-2k-1} < 2^{-2j} < \|x - y\|$. Also, $[x, u] \subset U_j$, $[u, v] \subset U_0$, and $[v, y] \subset U_k$. The other possibilities for positions of x, y can be dealt with similarly.

□



Remark 22. We close the paper with the last remark: if U is an open convex subset of a normed linear space X that contains an unbounded cone, i.e. the set $u + \bigcup_{t\in(0,+\infty)} tU(a,r)$ for some $u, a \in X$ and $r > 0$, then U has the $\bigl(1, 2\frac{\|a\|+r}{r}\bigr)$-UCC property. Indeed, without loss of generality we may assume that $u = 0$. Now for any $x, y \in U$ choose $R > 0$ such that $x, y \in U(0, R)$ and put $V = U(0, R) \cap U$. Since V clearly contains the ball $U\bigl(\frac{R}{\|a\|+r}a, \frac{R}{\|a\|+r}r\bigr)$, it follows that $e_V \le \frac{2R}{\frac{R}{\|a\|+r}r} = 2\frac{\|a\|+r}{r}$.

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