A SIMPLER PROOF OF THE APPROXIMATION BY REAL ANALYTIC LIPSCHITZ FUNCTIONS

MICHAL JOHANIS

Abstract. A theorem in [AFK] asserts that on a real separable Banach space with separating polynomial every Lipschitz function can be uniformly approximated by real analytic Lipschitz function with a control over the Lipschitz constant. We give a simpler proof of this theorem.

Using ideas from [K], [F], and [H] we give a simpler proof of the following theorem from [AFK].

Theorem 1 (Azagra-Fry-Keener). Let $X$ be a real separable Banach space with a separating polynomial. Then there is a constant $K \in \mathbb{R}$ such that for each $\varepsilon > 0$ and any $L$-Lipschitz function $f : X \to \mathbb{R}$ there is a $KL$-Lipschitz function $g \in C^\omega(X)$ satisfying $\sup_{x \in X} |f(x) - g(x)| \leq \varepsilon$.

By $U(x, r)$ (resp. $U(x, r)$) we denote the closed (resp. open) ball centred at $x$ with radius $r > 0$. If we need to stress that the ball is taken in the space $X$ we write $U_X(x, r)$. By $\tilde{X}$ we denote the Taylor complexification of a real Banach space $X$. By $H(\Omega)$ we denote the set of all holomorphic functions defined on an open subset $\Omega$ of a complex Banach space.

The proof is divided into a few steps (Proposition 2, Proposition 3, and Lemma 6). We begin by introducing an auxiliary notion.

Denote $h : \bigcup_{x \in \mathbb{R}} O_X(x) \to \mathbb{C}$ separates $A$ with respect to $U$ if

- $h(x) \to 1$ whenever $x \to A$,
- $|h(x)| \leq \frac{1}{2}$ whenever $x \to U_X$, $x \to X$, dist$x, A \geq 1$.

Proposition 2. Let $X$ be a real Banach space. Assume that there is $\mathcal{U} = \{U_x : x \in U_X \subseteq \tilde{X}, x \in X\}$ a collection of open neighbourhoods in $\tilde{X}$ and $C > 0$ such that for each $A \subseteq \tilde{X}$ there is a function $h_A \in H(\bigcup \mathcal{U})$ that separates $A$ with respect to $U$ and such that $h_A|_X$ is $C$-Lipschitz. Then for every $\varepsilon > 0$ and every $L$-Lipschitz function $f : X \to \mathbb{R}$ there is a $10CL$-Lipschitz function $g \in C^\omega(X)$ satisfying $\sup_{x \in X} |f(x) - g(x)| \leq \varepsilon$.

For the proof we need the following technical lemma.

Lemma 3. There are functions $\theta_n \in H(C)$, $n \in \mathbb{N}$, with the following properties:

- $(T1)$ $\theta_n \upharpoonright \mathbb{R}$ maps into $[0, 1]$,
- $(T2)$ $\theta_n \upharpoonright \mathbb{R}$ is $4$-Lipschitz,
- $(T3)$ $|\theta_n(z)| \leq 2^{-n}$ for every $z \in \mathbb{C}$, $|z| \leq \frac{1}{2}$,
- $(T4)$ $|\theta_n(x) - 1| \leq 2^{-n}$ for every $x \in N, x \geq 1$,
- $(T5)$ $|\theta_n(x)| \leq 2^{-n}$ for every $x \in \mathbb{R}$, $x \leq \frac{1}{2}$ or $x \geq 1$.

Proof of Proposition 2. Let us define a function $\hat{f} : X \to \mathbb{R}$ by $\hat{f}(x) = \frac{4}{r} f \left( \frac{x}{4 \mathbb{R}} \right)$. This function is obviously $1$-Lipschitz. Denote $\hat{f}^+ = \max\{\hat{f}, 0\}$ and $\hat{f}^- = \max\{-\hat{f}, 0\}$ and notice that both functions are again $1$-Lipschitz. Next, let us define sets $A_n = \{x \in X : \hat{f}^+(x) \geq n\}$ for $n \in \mathbb{N} \cup \{0\}$. Clearly, $A_n \subseteq A_{n-1}$ for all $n \in \mathbb{N}$, and using the $1$-Lipschitz property of $\hat{f}^+$ it is easy to check that

$$\text{dist}(X \setminus A_n, A_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \quad (1)$$

Denote $h_n(z) = \theta_n \circ h_{A_n}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $h_n \in H(\bigcup \mathcal{U})$ and $h_n|_X$ is $4C$-Lipschitz. Put $h^+ = \sum_{n=1}^{+\infty} h_n$.

Fix an arbitrary $x \in X$. Then there is $m \in \mathbb{N}$ such that $x \in A_{m-1} \setminus A_m$. Hence

$$x \in A_n \quad \text{for } n < m \quad \text{and} \quad x \in X \setminus A_{n-1} \quad \text{for } n > m. \quad (2)$$

From this, $(T1)$, $(S3)$, and $(T3)$ it follows that $|h_n(z)| \leq 2^{-n}$ for all $n \geq m$ and $z \in U_x$. Hence the sum in the definition of $h^+$ converges absolutely uniformly on $U_x$ and so $h^+ \in H(\bigcup \mathcal{U})$. This together with $(S1)$ and $(T1)$ implies that $h^+|_X \in C^\omega(X)$.

Using $(T2)$, $(S3)$, and $(T3)$, $(S3)$, $(T3)$ and $(T3)$, and finally $m - 1 + m \in [m - 1, m]$ and $f^+(x) \in [m - 1, m)$, we obtain

$$|h^+(x) - \hat{f}^+(x)| = \left| \sum_{n=1}^{m-1} h_n(x) + h_m(x) + \sum_{n=m+1}^{+\infty} h_n(x) - \hat{f}^+(x) \right| \leq \sum_{n=1}^{m-1} \left| h_n(x) - 1 \right| + \sum_{n=m+1}^{+\infty} \left| h_n(x) \right| + \left| m - 1 + h_m(x) - \hat{f}^+(x) \right| < \sum_{n=1}^{m-1} 2^{-n} + \sum_{n=m+1}^{+\infty} 2^{-n} + 1 < 2.$$
Further, from (1) it follows that there is a neighbourhood $U$ of $x$ in $X$ such that $U \subset X \setminus A_m$ and $U \subset A_n$ for $n < m - 1$. Thus $\|h_{A_n}(y)\| \leq \frac{1}{2}$ for $n > m$ and $y \in U$, and $h_{A_n}(y) \geq 1$ for $n < m - 1$ and $y \in U$. This together with (15) implies $\|h_n \cdot x\| \leq \frac{m}{m-1}C$ for $n \in \{m-1, m\}$ and $y \in U$. Hence $\sum_{n=1}^{\infty} \|h_n \cdot x\|$ converges absolutely uniformly on $U$ and so

$$\| (h^+ \cdot x)(x) \| \leq \sum_{n=1}^{\infty} \| (h_n \cdot x)(x) \| \leq \sum_{n \neq m} 2^{-n}C + \| (h_m \cdot x)(x) \| \leq C + 4C = 5C.$$  

Similarly we obtain an approximation of $\hat{f}^+$ denoted by $\hat{h}^-$. Put $h = h^+ - \hat{h}^-$. Then $h \cdot x \in C^m(X)$, $|h(x) - \hat{f}(x)| < 4$ for every $x \in X$, and $\| (h \cdot x)(x) \| \leq \| (h^+ \cdot x)(x) \| + \| (\hat{h}^- \cdot x)(x) \| < 10C$ for every $x \in X$.

Finally, let $g(x) = \frac{h}{4} (\frac{4L}{\varepsilon} x)$ for $x \in X$. It is straightforward to check that $g$ satisfies the conclusion of our proposition.

Let $X$ be a set. A collection $\{\psi_\alpha\}_{\alpha \in A}$ of functions on $X$ is called a supremal partition (sup-partition) if

- $\psi_\alpha : X \to [0, 1]$ for all $\alpha \in A$,
- there is a $Q > 0$ such that $sup_{\alpha \in A} \psi_\alpha (x) \geq Q$ for each $x \in X$,
- for each $x \in X$ and for each $\varepsilon > 0$ the set $\{ \alpha \in A : \psi_\alpha (x) > \varepsilon \}$ is finite.

**Proposition 4.** Let $X$ be a real Banach space. Suppose that there is an open neighbourhood $\hat{G}$ of $X$ in $\hat{X}$ and a collection $\{\psi_n\}_{n \in \mathbb{N}}$ of functions on $\hat{G}$ with the following properties:

- (P1) $\{\psi_n \cdot x\}_{n \in \mathbb{N}}$ is a sup-partition on $X$,
- (P2) the mapping $z \mapsto (a_n \psi_n(z))_{n \in \mathbb{N}}$ is a holomorphic mapping from $\hat{G}$ into $\mathcal{C}_0$ for any $(a_n) \in \ell_\infty$,
- (P3) there is $M > 0$ such that each $\psi_n \cdot x \in M$-Lipschitz,
- (P4) there is $r > 0$ such that for each $n \in \mathbb{N}$ there is $\delta_n > 0$ such that $\psi_n (x) \leq \frac{Q}{r}$ for $x \in X$, $|x - \delta_n| \geq r$.

Then there is a collection $\mathcal{U}$ of open neighbourhoods in $X$ such that for each $A \subset X$ there is a function $h_A \in H(\bigcup \mathcal{U})$ which separates $A$ with respect to $\mathcal{U}$ and such that $h_A \cdot x$ is $C$-Lipschitz, where $C = 2r \sqrt{2M/q}$.

In the proof we use the following proposition.

**Proposition 5.** Let $q \geq 1$. There is an open set $W \subset \mathbb{C}$ and a function $\mu \in H(W)$ with the following properties:

- (M1) For every $w \in c_0 \setminus \{0\}$ there is $\Delta_w > 0$ such that $U_{c_0}(y, \Delta_w) \subset W$ for every $y \in c_0$ satisfying $|w| \leq |y| \leq q |w|$, where the inequalities are understood in the lattice sense.
- (M2) $\mu(w) \geq 8$ for $w \in c_0 \setminus \{0\}$.
- (M3) $\mu(z) < 2$ for $z \in U_{c_0}(y, \Delta_w)$, where $\gamma \in c_0$, $\| \gamma \| \leq 1$, and $w \in c_0 \setminus \{0\}$.
- (M4) $\mu \cdot c_0$ is $\sqrt{2}$-Lipschitz and maps into $\mathbb{R}$.

**Proof of Proposition 5.** Let $W$, $\mu$, and $\Delta_w$ be as in Proposition 5 for $q = \frac{8}{Q}$. Further, we put $G = \frac{1}{3\sqrt{2}} \hat{G}$, $x_n = \frac{5}{2n}$, and $\psi_n(z) = \psi_n(2rz)$ for $z \in G$. Then the functions $\psi_n \cdot x$ are $2rM$-Lipschitz and

$$\psi_n(x) \leq \frac{Q}{8} \quad \text{for} \quad x \in X, \quad |x - x_n| \geq \frac{1}{2}.$$  

(3)

Denote $w(z) = (\psi_n(z))_{n \in \mathbb{N}}$ for $z \in G$. By the continuity of the mapping $w$ which follows from (P2), for each $x \in X$ there is an open neighbourhood $U_x$ of $x$ in $X$ such that $U_x \subset G$ and $\|w(z) - w(x)\| < \Delta_{w(x)}/q$ whenever $z \in U_x$. (Notice that $w(x) \in c_0 \setminus \{0\}$.) Put $\mathcal{U} = \{U_x : x \in X\}$.

Let $A \subset X$. For each $n \in \mathbb{N}$ put $b_n = q$ if $dist(x_n, A) \leq \frac{1}{2}$ and $b_n = 1$ otherwise. Choose $z \in \bigcup \mathcal{U}$ and let $x \in \mathcal{U}$ be such that $z \in U_x$. Then

$$\| (b_n \psi_n(z)) - (b_n \psi_n(x)) \| \leq sup_{n \in \mathbb{N}} \|b_n (\psi_n(z) - \psi_n(x))\| \leq q sup_{n \in \mathbb{N}} \|\psi_n(z) - \psi_n(x)\| = q \|w(z) - w(x)\| < \Delta_{w(x)}$$  

(4)

and since $0 \leq w(x) \leq (b_n \psi_n(x)) \leq q w(x)$ in the lattice sense, from (M1) it follows that $(b_n \psi_n(z)) \in W$. Therefore we may define $h_A(z) = \frac{1}{3} \mu((b_n \psi_n(z)))$ for $z \in \mathcal{U}$ and (P2) implies that $h_A \in H(\mathcal{U})$. Further, $h_A \cdot x$ is obviously $C$-Lipschitz.

Next we show that $h_A$ separates $A$ with respect to $\mathcal{U}$. Clearly $h_A$ has property (3). Pick any $x \in A$. From (4) and (5) it follows that $sup \{\psi_n(x) : n \in \mathbb{N}, dist(x_n, A) \leq \frac{1}{2} \} \geq Q$. Therefore $\| (b_n \psi_n(x)) \| \geq q Q = 8$ and consequently (M2) gives property (4).

Finally, to show (5) let $x \in X$ be such that $dist(x, A) \geq 1$. Then, by (3), $\psi_n(x) \leq \frac{Q}{8}$ for those $n \in \mathbb{N}$ for which $dist(x_n, A) \leq \frac{1}{2}$.

Thus $\| (b_n \psi_n(x)) \| \leq max \{q \frac{Q}{8}, 1 \} = 1$. Now (4) together with (M3) implies $|h_A(z)| \leq \frac{1}{2}$ for $z \in U_x$.

The following lemma finishes the proof of Theorem 1.

**Lemma 6.** Let $X$ be a real separable Banach space with a separating polynomial. Then there is an open neighbourhood $\hat{G}$ of $X$ in $\hat{X}$ and a collection of functions $\{\psi_n\}_{n \in \mathbb{N}}$ satisfying the properties (17)–(24) in Proposition 4. To prove this lemma we will need a few auxiliary statements.
Lemma 7. Let $X$ be a real Banach space with a separating polynomial. Then there is $\Delta > 0$ and a function $v \in H(\Omega)$, where $\Omega = \{x + iy \in \mathbb{X} : x, y \in X, |y| < \Delta\}$, such that $v|_X$ is Lipschitz and maps into $[0, +\infty)$, $v(0) = 0$, $v(x) \geq \|x\| - 1$ for $x \in X$, and the family of functions $\{y \mapsto \text{Im} v(x + iy) : y \in X, \|y\| < \Delta\}_{x \in X}$ is equicontinuous at 0.

Proof. It is an easy well-known fact that if $X$ admits a separating polynomial then $X$ admits a homogeneous separating polynomial (see e.g. [FPWZ]). Put $v(z) = (1 + P(z))^{1/n} - 1$ for a suitable $n$-homogeneous separating polynomial $P$. The equicontinuity follows from the fact that $v$ is even Lipschitz on the whole of $\Omega$. For the details see [AFK] Lemma 2.

\square

Lemma 8. There are functions $\phi_n \in H(C^n)$ and constants $\delta_n > 0$, $n \in \mathbb{N}$, with the following properties:

(H1) $\phi_n|_{\mathbb{X}}$ maps into $[0, 1]$.

(H2) $\phi_n|_{\mathbb{X}}$ is 1-Lipschitz with respect to the maximum norm.

(H3) $|\phi_n(z)| \leq 2^{-n}$ for every $z \in C^n$ such that there is $j \in \{1, \ldots, n - 1\}$ for which $\text{Re} z_j \leq \frac{1}{2}$ and $|\text{Im} z_j| \leq \delta_j$ for $i = 1, \ldots, n$.

(H4) $\phi_n(x) \geq \frac{1}{2}$ for every $x \in \mathbb{R}^n$ such that $x_n \leq 3$ and $x_i \geq 3$, $i = 1, \ldots, n - 1$.

(H5) $\phi_n(x) \leq \frac{1}{12}$ for $x \in \mathbb{R}^n$, $x_n \geq 5$.

With the aid of the statements above the proof of Lemma 9 is not difficult.

Proof of Lemma [5]. Let $v$ and $\omega$ be the function and the set from Lemma [7] and $\phi_n$ be the functions from Lemma [8] Let $\{x_n\}_{n \in \mathbb{N}}$ be a dense subset of $X$. Put

$$\psi_n(z) = \phi_n(v(z - x_1), \ldots, v(z - x_n))$$

for $z \in \omega, n \in \mathbb{N}$. Then $\psi_n \in H(\omega)$ and by (H6) $\psi_n|_{\mathbb{X}}$ maps into $[0, 1]$.

Let $M > 0$ be such that $\|v\| \leq M$. Pick any $x \in X$. Then from the density of the $\{x_n\}$ and the fact that $v(y) \leq M \|y\|$ for any $y \in X$ it follows that there is $k \in \mathbb{N}$ such that $\|v(x - x_k)\| \leq 3$. Let $k \in \mathbb{N}$ be the smallest such number. Then property (H4) implies that $\phi_k(x) \geq \frac{1}{2}$. Thus $\sup_{n \in \mathbb{N}} \psi_n(x) \geq \frac{M}{2}$ for each $x \in X$, where $Q = \frac{M}{2}$.

By the continuity of $v$ there is $\rho > 0$ such that $|v(z)| < \frac{1}{2}$ whenever $z \in \bar{X}, \|z\| < \rho$. Now fix $x \in X$ and find an index $j \in \mathbb{N}$ such that $\|x_j - x_k\| < \rho$. Using the equicontinuity of $\{y \mapsto \text{Im} v(w + iy)\}$, $0 \leq \Delta_j < \Delta$ such that $|\text{Im} v(w + iy)| < \delta_j$ whenever $w, y \in X, \|y\| < \Delta_j$. Let us define $U_x = \{z = x + iy \in \mathbb{X} : \|x\| < \rho, \|y\| < \Delta_j\}$. Notice that $U_x$ is an open neighbourhood of $x$ and $x_j - x_k \in \Omega$ for every $x \in U_x, l \in \mathbb{N}$. Let $z = w + iy \in U_x$. Then $|\text{Im} v(z - x_l)| = |\text{Im} v(w + x_l - x_k + iy)| < \delta_j$ for every $l \in \mathbb{N}$ and $|\text{Im} v(z - x_l)| \leq |\text{Im} v(z - x_l)| < \frac{1}{2}$. Hence, by (H3), $|\phi_n(z)| \leq 2^{-n}$ for $n > j$ and $z \in U_x$. It follows that for any $a_n \in \mathbb{R}, \sum_{n=1}^{\infty} a_n \phi_n(z) \in c_0$ and the sum converges absolutely uniformly on $U_x$. As the mappings $\text{Re} a_n \phi_n(z)$ are holomorphic as mappings from $U_x$ into $c_0$, we can conclude that $\sum_{n=1}^{\infty} a_n \phi_n(z) \in c_0$. This function is holomorphic on $U_x$ and $\psi_n(x) = \sum_{n=1}^{\infty} a_n \phi_n(z) \in c_0$ which gives (H2).

Property (H3) obviously holds by (H2). Finally we show that (H5) is satisfied with $r = 6$. Indeed, fix $n \in \mathbb{N}$. For $x \in X, \|x - x_n\| \geq 6$ we have $v(x - x_n) \geq \|x - x_n\| - 1 \geq 5$, hence, by (H5), $\psi_n(x) \leq \frac{1}{12} \leq \frac{2}{3}$.

\square

For the proof of Proposition [5] we need the following version of the Implicit Function Theorem with explicit estimates on the size of the region where the solution is found.

Theorem 9 (Implicit Function Theorem). Let $X$ be a complex Banach space, $X \subset X$ and $V \subset \mathbb{C}$ open sets, and $F \subset H(U \times V)$. Let $\zeta \in U$, $\eta \in V$ satisfy $F(\zeta, \eta) = 0$. Further let $R > 0$, $S > 0$, and $M > 0$ be such that $B(\zeta, S) \subset U$, $B(\eta, R) \subset V$, and $F(z, u) \leq M$ for every $z \in B(S, S), u \in B(\eta, R)$. Assume that $|\frac{\partial F}{\partial \tilde{\eta}}(\zeta, \tilde{\eta})| \geq a > 0$ and $0 < r < aR^2 - \frac{\|F\|_M}{2}$. Put $c = ar - \frac{aR^2}{2}$ and $s = \sqrt{\frac{2}{M}}$. Then for each $z \in U(\zeta, s)$ there is a unique $u \in U(\eta, r)$ satisfying $F(z, u) = 0$. Denote such $u$ by $\varphi(z)$. Then $\varphi \subset H(U(\zeta, s))$.

The proof of this theorem is fairly standard using for example the Rouché theorem and Cauchy’s estimates for $\frac{\partial F}{\partial \tilde{\eta}}$ on $C$ and for $\frac{\partial F}{\partial \tilde{\eta}}$ on $X$. Some details can be found e.g. in [CHP]. Some estimates and the proofs given there are not entirely correct.

Proof of Proposition [5]. Define $\mu$ on $c_0$ as the Minkowski functional of the set $\{x \in c_0 : \sum_{n=1}^{\infty} (x_n)^2 \leq 1\}$. Then $\mu$ is an equivalent norm on $c_0$ for which $\|x\| \leq \mu(x) \leq \sqrt{2} \|x\|$ (see [FPWZ]). This gives property (M1) and (M2).

Let $F : \mathbb{C} \times (C \setminus \{0\}) \to C$ be defined as $F(z, u) = \sum_{n=1}^{\infty} (z_n u_n)^2 - 1$. This function is holomorphic on $c_0 \times (C \setminus \{0\})$ and for every $x \in c_0 \setminus \{0\}$ we have $F(x, v) = 0$.

Fix $w \in c_0 \setminus \{0\}$. Put $R = \frac{\|w\|}{2}$. Let $M = 1 + \sum_{n=1}^{\infty} \left( \frac{1}{2} + \frac{aR^2}{2\|F\|_M} \right)^2 \frac{1}{\|w\|^2} - 1$, $a = \frac{1}{\|w\|^2} - 1$, and $\Delta_w = s$ as defined in Theorem 9. Now choose any $y \in c_0, |w| \leq |y| \leq |w|$. Then $R < |y| \leq |w| \leq |y| \leq \mu(y)$, and $B(\mu(y), R) \subset C \setminus \{0\}$. For any $z \in B(y, S), u \in B(\mu(y), R)$ we have $|w| \geq \mu(y) \geq |y| - |w| - R < |w|$. Finally,\[ |\frac{\partial F}{\partial \tilde{\eta}}(y, \mu(y))| = \frac{1}{\|w\|^2} \sum_{n=1}^{\infty} (w_n u_n)^2 \geq \frac{1}{\|w\|^2} \sum_{n=1}^{\infty} (\frac{1}{|w|} u_n)^2 \geq \frac{1}{\|w\|^2} \geq a \]. Thus by Theorem 9 the equation $F(z, u) = 0$ uniquely determines a holomorphic function $\mu_y^w$ on $U(c_0, \Delta_w)$ with values in $U(\mu(y), r)$ and this holds for every $y \in c_0, |w| \leq |y| \leq |w|$.\]
Take any two functions $\mu_1 = \mu_{\gamma_1}^{w_1}, \mu_2 = \mu_{\gamma_2}^{w_2}$ defined on open balls $U_1$ and $U_2$ respectively. If $U_1$ and $U_2$ intersect, then it is easy to check that $U_1 \cap U_2 \cap c_0$ is a non-empty set relatively open in $c_0$. Since $\mu_1 = \mu$ on $U_1 \cap c_0$ and $\mu_2 = \mu$ on $U_2 \cap c_0$, it follows that both holomorphic functions $\mu_1$ and $\mu_2$ agree on some ball in $U_1 \cap U_2$ and therefore on the whole $U_1 \cap U_2$. This observation allows us to put $W = \bigcup \{ U_{\gamma}(y, \Delta w) : y \in c_0, \|y\| \leq \|w\| \}$ and define $\mu$ on $W$ by $\mu(z) = \mu_{\gamma}^{w}(z)$ whenever $z \in U(y, \Delta w)$. This gives property (M1).

To prove (M3) let $w \in c_0 \setminus \{0\}$, $y \in c_0$, $|w| \leq |y| \leq q|w|$, $\|y\| \leq 1$, and $z \in U_{\gamma_{\gamma}(y, \Delta w)}$. Then by the choice of $r$ above we have $\mu(z) \in U(\mu(y), 2 - \sqrt{2})$ and therefore $|\mu(z)| < |\mu(y)| + 2 - \sqrt{2} \leq 2 \|y\| + 2 - \sqrt{2} \leq 2$.

\[
\square
\]

It remains to prove Lemma 3 and Lemma 8. The proofs are standard using integral convolution technique and estimates which are not difficult. We could just write the formulas in consideration and stop there (we claim a short proof after all). Nevertheless for the convenience of the reader we include rather detailed computations.

**Proof of Lemma 8.** Let $\zeta_n : \mathbb{R}^n \to [0, 1]$ be a 1-Lipschitz function (with respect to the maximum norm) such that

\[
\zeta_n(x) = \begin{cases} 
0 & \text{whenever } x_n \geq 4 \text{ or } \exists i \in \{1, \ldots, n-1\} : x_i \leq 1, \\
1 & \text{whenever } x_n \leq 3 \text{ and } \forall i \in \{1, \ldots, n-1\} : x_i \geq 2.
\end{cases}
\]

For each $n \in \mathbb{N}$ put $d_n = \sqrt{2^{-n}/8}$ and find $a_n \in \mathbb{R}$ such that

\[
a_n 2^{-n} \geq 3, \quad e^{-an 2^{-n}/8} \leq 2 \sqrt{\pi} \cdot 2^{-n}, \quad \int_{-\sqrt{an 2^{-n}}}^{+\infty} e^{-t^2} dt \geq \frac{1}{\sqrt{2} \sqrt{\pi}}.
\]

Finally, put

\[
\phi_n(z) = \frac{1}{c_n} \int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum_{i=1}^{n} (z_i - t_i)^2} dt \quad \text{for } z \in \mathbb{C}^n,
\]

where $c_n = \int_{\mathbb{R}^n} e^{-an \sum_{i=1}^{n} t_i^2} dt = \sqrt{\frac{(2\pi)^n}{an}} \prod_{i=1}^{n} \Gamma(1/2)$.

Using standard theories on integrals dependent on parameter we obtain $\phi_n \in H(\mathbb{C}^n)$. Property (H1) is obvious, and property (H2) is easy to check.

Next we will need the elementary estimate

\[
\int_{x}^{+\infty} e^{-t^2} dt \leq \int_{x}^{+\infty} te^{-t^2} dt = \frac{1}{2} e^{-x^2} \quad \text{for } x \geq 1.
\]

To prove (H3) we use successively the definition of $\zeta_n$, Fubini’s theorem, substitution, $\text{Re } z_j \leq \frac{1}{2}$, estimate (6) together with (5), the definition of $\delta_j$, and finally (6) to obtain

\[
|\phi_n(z)| \leq \frac{1}{c_n} \int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum_{i=1}^{n} (z_i - t_i)^2} dt = \frac{a_n}{c_n} \int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum_{i=1}^{n} (z_i - t_i)^2} dt
\]

\[
\leq \frac{a_n \delta_j}{\sqrt{\pi}} \int_{j_1 > 1} \int_{j_n}^{+\infty} e^{-an 2^{-j} (Re z_j - t_j)^2} dt_j = \frac{a_n \delta_j}{\sqrt{\pi}} \int_{j_1 > 1} \int_{j_n}^{+\infty} e^{-an 2^{-j} (Re z_j - t_j)^2} dt_j \leq \frac{a_n \delta_j}{\sqrt{\pi}} \int_{j_1 > 1} \int_{j_n}^{+\infty} e^{-an 2^{-j} (Re z_j - t_j)^2} dt_j
\]

\[
\leq \frac{a_n \delta_j}{2 \sqrt{\pi}} \cdot e^{-4a_n 2^{-j/4} \delta_j/\delta_j} \leq \frac{1}{2 \sqrt{\pi}} \cdot e^{-an 2^{-n/8}} \leq \frac{1}{2 \sqrt{\pi}} \cdot e^{-an 2^{-n/8}} \leq 2^{-n}.
\]

To prove (H4) we use successively the definition of $\zeta_n$, Fubini’s theorem and substitution, $x_n \leq 3$ and $x_i \geq 3$, substitution, and (7) to obtain

\[
\phi_n(x) \geq \frac{1}{c_n} \int_{t_1 \geq 2, \ldots, t_n \leq 3} e^{-a_n \sum_{i=1}^{n} (x_i - t_i)^2} dt \geq \frac{1}{2} \int_{-\infty}^{+\infty} e^{-an 2^{-n} t_i^2} dt \int_{-\infty}^{+\infty} e^{-an 2^{-n} t_i^2} dt \geq \frac{1}{2} \int_{-\infty}^{+\infty} e^{-an 2^{-n} t_i^2} dt \geq \frac{1}{2} \int_{-\infty}^{+\infty} e^{-an 2^{-n} t_i^2} dt \geq \frac{1}{2} \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n \geq \frac{1}{4}.
\]
Finally, to prove (15) we use successively the definition of $\zeta_n$, Fubini’s theorem, substitution, $x_n \geq 5$, and (8) together with (5) to obtain
\[
\phi_n(x) \leq \frac{1}{c_n} \int_{a_n^{-\frac{1}{2}}}^{a_n^{\frac{1}{2}}} e^{-\frac{n}{2}(x_n-t)^2} \, dt = \frac{\sqrt{\pi}}{c_n} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(x_n-t)^2} \, dt = \frac{1}{c_n} \sqrt{\frac{\pi}{a_n}} e^{-\frac{n}{2}(x_n-t)^2} \, dt \leq \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2} \, dt \leq \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{n}{2}(x_n-t)^2} \leq \frac{1}{32}.
\]

\[\square\]

**Proof of Lemma [3]** Let $\xi : \mathbb{R} \to [0, 1]$ be defined as $\xi(t) = 0$ for $t \leq \frac{5}{8}$, $\xi(t) = 4t - \frac{5}{2}$ for $t \in (\frac{5}{8}, \frac{7}{8})$, and $\xi(t) = 1$ for $t \geq \frac{7}{8}$. Obviously $\xi$ is a 4-Lipschitz function. For each $n \in \mathbb{N}$ find $a_n \in \mathbb{R}$ such that
\[
\frac{3}{8} \sqrt{a_n} \geq 1,
\]
\[
e^{-\frac{5}{4}a_n} \leq 2\sqrt{\pi} \cdot 2^{-n},
\]
\[
\int_{-\frac{1}{\sqrt{a_n}}}^{+\infty} e^{-t^2} \, dt \geq (1 - 2^{-n}) \sqrt{\pi}, \quad \text{and}
\]
\[
2\sqrt{a_n} \cdot e^{-\frac{1}{\sqrt{a_n}}} \leq \sqrt{\pi} \cdot 2^{-n}.
\]

Finally, put
\[
\theta_n(z) = \frac{1}{c_n} \int_{\mathbb{R}} \xi(t)e^{-a_n(z-t)^2} \, dt \quad \text{for } z \in \mathbb{C},
\]
where $c_n = \int_{\mathbb{R}} e^{-a_n t^2} \, dt = \sqrt{\frac{\pi}{a_n}}$.

Using standard theorems on integrals dependent on parameter we obtain $\theta_n \in H(\mathbb{C})$. Property (11) is obvious, and property (12) is easy to check.

To prove (13) we use successively the definition of $\xi$, $|\text{Im } z| \leq \frac{1}{4}$, substitution, $\text{Re } z \leq \frac{1}{4}$, estimate (8) together with (9), and finally (10) to obtain
\[
\begin{align*}
|\theta_n(z)| &\leq \frac{1}{c_n} \int_{\mathbb{R}} \xi(t)e^{-a_n \text{Re}(z-t)^2} \, dt = \frac{e^{a_n \text{Re}(z)}}{c_n} \int_{\mathbb{R}} \xi(t)e^{-a_n (\text{Re} z - t)^2} \, dt \\
&= \frac{e^{a_n \text{Re}(z)}}{\sqrt{\pi}} \int_{\sqrt{a_n} \cdot (\text{Re} z)}^{+\infty} e^{-t^2} \, dt \leq \frac{e^{a_n \text{Re}(z)}}{\sqrt{\pi}} \cdot e^{-\frac{1}{\sqrt{a_n}}} \leq \frac{e^{-\frac{5}{4}a_n}}{2\sqrt{\pi}} \leq 2^{-n}.
\end{align*}
\]

To prove (14) we use successively the definition of $\xi$, substitution, $x \geq 1$, and (11) to obtain
\[
\theta_n(x) \geq \frac{1}{c_n} \int_{\frac{7}{8}}^{+\infty} e^{-a_n(x-t)^2} \, dt = \frac{1}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{a_n}}}^{+\infty} e^{-t^2} \, dt \geq \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{a_n}}}^{+\infty} e^{-t^2} \, dt \geq 1 - 2^{-n}.
\]

Finally, we show (15). Differentiating under the integral sign we obtain
\[
\theta_n(x) = \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} \xi(t)(t-x)e^{-a_n(t-x)^2} \, dt.
\]
Thus for $x \leq \frac{1}{4}$ using the definition of $\xi$, substitution, and (12) we get
\[
|\theta_n(x)| \leq \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} (t-x)e^{-a_n(t-x)^2} \, dt = \frac{1}{c_n} \int_{-\infty}^{\frac{7}{8}} e^{-a_n(x-y)^2} \, dy = \sqrt{\frac{\pi}{a_n}} e^{-a_n \frac{7}{8} x} \leq \sqrt{\frac{\pi}{a_n}} \cdot e^{-\frac{5}{4}a_n} \leq 2^{-n}.
\]

On the other hand, for $x \geq 1$ using the definition of $\xi$, evaluation of the integrals, and (12) we get
\[
|\theta_n(x)| = \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} \xi(t)(t-x)e^{-a_n(t-x)^2} \, dt + \int_{\frac{7}{8}}^{+\infty} (t-x)e^{-a_n(t-x)^2} \, dt \leq \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} \xi(t)(t-x)e^{-a_n(t-x)^2} \, dt + \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} (t-x)e^{-a_n(t-x)^2} \, dt \leq \frac{2a_n}{c_n} \int_{\frac{7}{8}}^{+\infty} \xi(t)(t-x)e^{-a_n(t-x)^2} \, dt + \frac{1}{c_n} \cdot e^{-a_n \frac{7}{8} x} \leq \frac{1}{c_n} \left(e^{-a_n \frac{7}{8} x} - e^{-a_n \frac{7}{8} x} + e^{-a_n \frac{7}{8} x}\right) \leq \frac{2}{c_n} \cdot e^{-a_n \frac{7}{8} x} \leq 2 \sqrt{\frac{\pi}{a_n}} \cdot e^{-\frac{5}{4}a_n} \leq 2^{-n}.
\]

\[\square\]
REFERENCES


DEPARTMENT OF MATHEMATICAL ANALYSIS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, 186 75 Praha 8, CZECH REPUBLIC
E-mail address: johanis@karlin.mff.cuni.cz