A SIMPLER PROOF OF THE APPROXIMATION BY REAL ANALYTIC LIPSCHITZ FUNCTIONS

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ABSTRACT. A theorem in [AFK] asserts that on a real separable Banach space with separating polynomial every Lipschitz function can be uniformly approximated by real analytic Lipschitz function with a control over the Lipschitz constant. We give a simpler proof of this theorem.

Using ideas from [K], [F], and [HJ] we give a simpler proof of the following theorem from [AFK].

Theorem 1 (Azagra-Fry-Keener)**.** *Let X be a real separable Banach space with a separating polynomial. Then there is a constant $K \in \mathbb{R}$ such that for each $\varepsilon > 0$ and any L-Lipschitz function $f: X \to \mathbb{R}$ there is a KL-Lipschitz function $g \in C^\omega(X)$ satisfying $\sup_{x\in X} |f(x) - g(x)| \leq \varepsilon$.*

By $B(x,r)$ (resp. $U(x,r)$) we denote the closed (resp. open) ball centred at x with radius $r > 0$. If we need to stress that the ball is taken in the space X we write $U_X(x,r)$. By $\tilde{X}$ we denote the Taylor complexification of a real Banach space X. By $H(\Omega)$ we denote the set of all holomorphic functions defined on an open subset Ω of a complex Banach space.

The proof is divided into a few steps (Proposition 2, Proposition 4, and Lemma 6). We begin by introducing an auxiliary notion. Let X be a real Banach space and $\mathcal{U} = \{U_x;\ x \in U_x \subset \tilde{X}, x \in X\}$ be a collection of open neighbourhoods in $\tilde{X}$. Let $A \subset X$. We say that a function $h: \bigcup \mathcal{U} \to \mathbb{C}$ *separates A with respect to $\mathcal{U}$* if

(S1) $h\restriction_X$ maps into $\mathbb{R}$,
(S2) $h(x) \geq 1$ whenever $x \in A$,
(S3) $|h(z)| \leq \frac{1}{4}$ whenever $z \in U_x$, $x \in X$, $\operatorname{dist}(x,A) \geq 1$.

Proposition 2. *Let X be a real Banach space. Assume that there is $\mathcal{U} = \{U_x;\ x \in U_x \subset \tilde{X}, x \in X\}$ a collection of open neighbourhoods in $\tilde{X}$ and $C > 0$ such that for each $A \subset X$ there is a function $h_A \in H\big(\bigcup \mathcal{U}\big)$ which separates A with respect to $\mathcal{U}$ and such that $h_A\restriction_X$ is C-Lipschitz. Then for every $\varepsilon > 0$ and every L-Lipschitz function $f: X \to \mathbb{R}$ there is a $10CL$-Lipschitz function $g \in C^\omega(X)$ satisfying $\sup_{x\in X} |f(x) - g(x)| \leq \varepsilon$.*

For the proof we need the following technical lemma.

Lemma 3. *There are functions $\theta_n \in H(\mathbb{C})$, $n \in \mathbb{N}$, with the following properties:*

(T1) $\theta_n\restriction_{\mathbb{R}}$ maps into $[0,1]$,
(T2) $\theta_n\restriction_{\mathbb{R}}$ is 4-Lipschitz,
(T3) $|\theta_n(z)| \leq 2^{-n}$ for every $z \in \mathbb{C}$, $|z| \leq \frac{1}{4}$,
(T4) $|\theta_n(x) - 1| \leq 2^{-n}$ for every $x \in \mathbb{R}$, $x \geq 1$,
(T5) $|(\theta_n\restriction_{\mathbb{R}})'(x)| \leq 2^{-n}$ for every $x \in \mathbb{R}$, $x \leq \frac{1}{2}$ or $x \geq 1$.

Proof of Proposition 2. Let us define a function $\hat{f}: X \to \mathbb{R}$ by $\hat{f}(x) = \frac{4}{\varepsilon} f\big(\frac{\varepsilon}{4L} x\big)$. This function is obviously 1-Lipschitz. Denote $\hat{f}^+ = \max\{\hat{f}, 0\}$ and $\hat{f}^- = \max\{-\hat{f}, 0\}$ and notice that both functions are again 1-Lipschitz. Next, let us define sets $A_n = \{x \in X;\ \hat{f}^+(x) \geq n\}$ for $n \in \mathbb{N} \cup \{0\}$. Clearly, $A_n \subset A_{n-1}$ for all $n \in \mathbb{N}$, and using the 1-Lipschitz property of $\hat{f}^+$ it is easy to check that

$$\operatorname{dist}(X \setminus A_n, A_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Denote $h_n(z) = \theta_n \circ h_{A_n}$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $h_n \in H\big(\bigcup \mathcal{U}\big)$ and $h_n\restriction_X$ is $4C$-Lipschitz. Put $h^+ = \sum_{n=1}^\infty h_n$.

Fix an arbitrary $x \in X$. Then there is $m \in \mathbb{N}$ such that $x \in A_{m-1} \setminus A_m$. Hence

$$x \in A_n \text{ for } n < m \quad \text{and} \quad x \in X \setminus A_{n-1} \text{ for } n > m. \tag{2}$$

From this, (1), (S3), and (T3) it follows that $|h_n(z)| \leq 2^{-n}$ for all $n > m$ and $z \in U_x$. Hence the sum in the definition of h^+ converges absolutely uniformly on U_x and so $h^+ \in H\big(\bigcup \mathcal{U}\big)$. This together with (S1) and (T1) implies that $h^+\restriction_X \in C^\omega(X)$.

Using (2), (S2) and (T4), (1), (S3) and (T3), and finally $m - 1 + h_m(x) \in [m-1, m]$ and $\hat{f}^+(x) \in [m-1, m)$, we obtain

$$\begin{aligned}
\big|h^+(x) - \hat{f}^+(x)\big| &= \left|\sum_{n=1}^{m-1} h_n(x) + h_m(x) + \sum_{n=m+1}^{\infty} h_n(x) - \hat{f}^+(x)\right| \\
&\leq \sum_{n=1}^{m-1} |h_n(x) - 1| + \sum_{n=m+1}^{\infty} |h_n(x)| + \big|m - 1 + h_m(x) - \hat{f}^+(x)\big| < \sum_{n=1}^{m-1} 2^{-n} + \sum_{n=m+1}^{\infty} 2^{-n} + 1 < 2.
\end{aligned}$$

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Further, from (1) it follows that there is a neighbourhood U of x in X such that $U \subset X \setminus A_m$ and $U \subset A_n$ for $n < m-1$. Thus $|h_{A_n}(y)| \leq \frac{1}{4}$ for $n > m$ and $y \in U$, and $h_{A_n}(y) \geq 1$ for $n < m-1$ and $y \in U$. This together with (T5) implies $\|(h_n\restriction_X)'(y)\| = |(\theta_n\restriction_{\mathbb{R}})'(h_{A_n}(y))|\,\|(h_{A_n}\restriction_X)'(y)\| \leq 2^{-n}C$ for $n \in \mathbb{N} \setminus \{m-1, m\}$ and $y \in U$. Hence $\sum_{n=1}^{\infty}(h_n\restriction_X)'$ converges absolutely uniformly on U and so

$$\left\|(h^+\restriction_X)'(x)\right\| \leq \sum_{n=1}^{\infty}\left\|(h_n\restriction_X)'(x)\right\| \leq \sum_{n\neq m} 2^{-n}C + \left\|(h_m\restriction_X)'(x)\right\| < C + 4C = 5C.$$

Similarly we obtain an approximation of $\hat{f}^-$ denoted by h^-. Put $h = h^+ - h^-$. Then $h\restriction_X \in C^{\omega}(X)$, $\left|h(x) - \hat{f}(x)\right| < 4$ for every $x \in X$, and $\|(h\restriction_X)'(x)\| \leq \|(h^+\restriction_X)'(x)\| + \|(h^-\restriction_X)'(x)\| < 10C$ for every $x \in X$.

Finally, let $g(x) = \frac{\varepsilon}{4}h\left(\frac{4L}{\varepsilon}x\right)$ for $x \in X$. It is straightforward to check that g satisfies the conclusion of our proposition.

□

Let X be a set. A collection $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of functions on X is called a *supremal partition (sup-partition)* if

- $\psi_\alpha\colon X \to [0,1]$ for all $\alpha \in \Lambda$,
- there is a $Q > 0$ such that $\sup_{\alpha\in\Lambda}\psi_\alpha(x) \geq Q$ for each $x \in X$,
- for each $x \in X$ and for each $\varepsilon > 0$ the set $\{\alpha \in \Lambda;\ \psi_\alpha(x) > \varepsilon\}$ is finite.

Proposition 4. *Let X be a real Banach space. Suppose that there is an open neighbourhood $\hat{G}$ of X in $\tilde{X}$ and a collection $\{\hat{\psi}_n\}_{n\in\mathbb{N}}$ of functions on $\hat{G}$ with the following properties:*

(P1) $\{\hat{\psi}_n\restriction_X\}_{n\in\mathbb{N}}$ is a sup-partition on X,
(P2) the mapping $z \mapsto (a_n\hat{\psi}_n(z))_{n\in\mathbb{N}}$ is a holomorphic mapping from $\hat{G}$ into $\tilde{c}_0$ for any $(a_n) \in \ell_\infty$,
(P3) there is $M > 0$ such that each $\hat{\psi}_n\restriction_X$ is M-Lipschitz,
(P4) there is $r > 0$ such that for each $n \in \mathbb{N}$ there is $\hat{x}_n \in X$ such that $\hat{\psi}_n(x) \leq \frac{Q}{8}$ for $x \in X$, $\|x - \hat{x}_n\| \geq r$.

Then there is a collection $\mathcal{U}$ of open neighbourhoods in $\tilde{X}$ such that for each $A \subset X$ there is a function $h_A \in H\left(\bigcup\mathcal{U}\right)$ which separates A with respect to $\mathcal{U}$ and such that $h_A\restriction_X$ is C-Lipschitz, where $C = 2r\sqrt{2}M/Q$.

In the proof we use the following proposition.

Proposition 5. *Let $q \geq 1$. There is an open set $W \subset \tilde{c}_0$ and a function $\mu \in H(W)$ with the following properties:*

(M1) For every $w \in c_0 \setminus \{0\}$ there is $\Delta_w > 0$ such that $U_{\tilde{c}_0}(y, \Delta_w) \subset W$ for every $y \in c_0$ satisfying $|w| \leq |y| \leq q\,|w|$, where the inequalities are understood in the lattice sense.
(M2) $\mu(w) \geq 8$ for $w \in c_0$, $\|w\| \geq 8$,
(M3) $|\mu(z)| < 2$ for $z \in U_{\tilde{c}_0}(y, \Delta_w)$, where $y \in c_0$, $\|y\| \leq 1$, and $w \in c_0 \setminus \{0\}$, $|w| \leq |y| \leq q\,|w|$,
(M4) $\mu\restriction_{c_0}$ is $\sqrt{2}$-Lipschitz and maps into $\mathbb{R}$.

Proof of Proposition 4. Let W, μ, and Δ_w be as in Proposition 5 for $q = \frac{8}{Q}$. Further, we put $G = \frac{1}{2r}\hat{G}$, $x_n = \frac{\hat{x}_n}{2r}$, and $\psi_n(z) = \hat{\psi}_n(2rz)$ for $z \in G$. Then the functions $\psi_n\restriction_X$ are $2rM$-Lipschitz and

$$\psi_n(x) \leq \frac{Q}{8} \quad \text{for } x \in X,\ \|x - x_n\| \geq \frac{1}{2}. \tag{3}$$

Denote $w(z) = (\psi_n(z))_{n\in\mathbb{N}}$ for $z \in G$. By the continuity of the mapping w (which follows from (P2)), for each $x \in X$ there is an open neighbourhood U_x of x in $\tilde{X}$ such that $U_x \subset G$ and $\|w(z) - w(x)\| < \Delta_{w(x)}/q$ whenever $z \in U_x$. (Notice that $w(x) \in c_0 \setminus \{0\}$.) Put $\mathcal{U} = \{U_x;\ x \in X\}$.

Let $A \subset X$. For each $n \in \mathbb{N}$ put $b_n = q$ if $\operatorname{dist}(x_n, A) \leq \frac{1}{2}$ and $b_n = 1$ otherwise. Choose $z \in \bigcup\mathcal{U}$ and let $x \in X$ be such that $z \in U_x$. Then

$$\|(b_n\psi_n(z)) - (b_n\psi_n(x))\| = \sup_{n\in\mathbb{N}}|b_n(\psi_n(z) - \psi_n(x))| \leq q\sup_{n\in\mathbb{N}}|\psi_n(z) - \psi_n(x)| = q\,\|w(z) - w(x)\| < \Delta_{w(x)} \tag{4}$$

and since $0 \leq w(x) \leq (b_n\psi_n(x)) \leq qw(x)$ in the lattice sense, from (M1) it follows that $(b_n\psi_n(z)) \in W$. Therefore we may define $h_A(z) = \frac{1}{8}\mu\big((b_n\psi_n(z))\big)$ for $z \in \bigcup\mathcal{U}$ and (P2) implies that $h_A \in H\left(\bigcup\mathcal{U}\right)$. Further, $h_A\restriction_X$ is obviously C-Lipschitz.

Next we show that h_A separates A with respect to $\mathcal{U}$. Clearly h_A has property (S1). Pick any $x \in A$. From (P1) and (3) it follows that $\sup\{\psi_n(x);\ n \in \mathbb{N}, \operatorname{dist}(x_n, A) \leq \frac{1}{2}\} \geq Q$. Therefore $\|(b_n\psi_n(x))\| \geq qQ = 8$ and consequently (M2) gives property (S2). Finally, to show (S3) let $x \in X$ be such that $\operatorname{dist}(x, A) \geq 1$. Then, by (3), $\psi_n(x) \leq \frac{Q}{8}$ for those $n \in \mathbb{N}$ for which $\operatorname{dist}(x_n, A) \leq \frac{1}{2}$. Thus $\|(b_n\psi_n(x))\| \leq \max\left\{q\frac{Q}{8}, 1\right\} = 1$. Now (4) together with (M3) implies $|h_A(z)| \leq \frac{1}{4}$ for $z \in U_x$.

□

The following lemma finishes the proof of Theorem 1.

Lemma 6. *Let X be a real separable Banach space with a separating polynomial. Then there is an open neighbourhood G of X in $\tilde{X}$ and a collection of functions $\{\psi_n\}_{n\in\mathbb{N}}$ satisfying the properties (P1)–(P4) in Proposition 4.*

To prove this lemma we will need a few auxiliary statements.

Lemma 7. *Let X be a real Banach space with a separating polynomial. Then there is $\Delta > 0$ and a function $\nu \in H(\Omega)$, where $\Omega = \{x + iy \in \tilde{X};\ x, y \in X, \|y\| < \Delta\}$, such that $\nu \restriction_X$ is Lipschitz and maps into $[0, +\infty)$, $\nu(0) = 0$, $\nu(x) \geq \|x\| - 1$ for $x \in X$, and the family of functions $\{y \mapsto \operatorname{Im} \nu(x + iy);\ y \in X, \|y\| < \Delta\}_{x \in X}$ is equicontinuous at 0.*

Proof. It is an easy well-known fact that if X admits a separating polynomial then X admits a homogeneous separating polynomial (see e.g. [FPWZ]). Put $\nu(z) = (1 + P(z))^{1/n} - 1$ for a suitable n-homogeneous separating polynomial P. The equicontinuity follows from the fact that ν is even Lipschitz on the whole of Ω. For the details see [AFK, Lemma 2].

□

Lemma 8. *There are functions $\phi_n \in H(\mathbb{C}^n)$ and constants $\delta_n > 0$, $n \in \mathbb{N}$, with the following properties:*

(H1) $\phi_n \restriction_{\mathbb{R}^n}$ *maps into* $[0, 1]$,
(H2) $\phi_n \restriction_{\mathbb{R}^n}$ *is* 1*-Lipschitz with respect to the maximum norm,*
(H3) $|\phi_n(z)| \leq 2^{-n}$ *for every* $z \in \mathbb{C}^n$ *such that there is* $j \in \{1, \dots, n-1\}$ *for which* $\operatorname{Re} z_j \leq \frac{1}{2}$ *and* $|\operatorname{Im} z_i| \leq \delta_j$ *for* $i = 1, \dots, n$,
(H4) $\phi_n(x) \geq \frac{1}{4}$ *for every* $x \in \mathbb{R}^n$ *for which* $x_n \leq 3$ *and* $x_i \geq 3$, $i = 1, \dots, n-1$,
(H5) $\phi_n(x) \leq \frac{1}{32}$ *for* $x \in \mathbb{R}^n$, $x_n \geq 5$.

With the aid of the statements above the proof of Lemma 6 is not difficult.

Proof of Lemma 6. Let ν and Ω be the function and the set from Lemma 7 and ϕ_n be the functions from Lemma 8. Let $\{x_n\}_{n \in \mathbb{N}}$ be a dense subset of X. Put

$$\psi_n(z) = \phi_n\big(\nu(z - x_1), \dots, \nu(z - x_n)\big) \qquad \text{for } z \in \Omega, n \in \mathbb{N}.$$

Then $\psi_n \in H(\Omega)$ and by (H1) $\psi_n \restriction_X$ maps into $[0, 1]$.

Let $M > 0$ be such that $\nu \restriction_X$ is M-Lipschitz. Pick any $x \in X$. Then from the density of $\{x_n\}$ and the fact that $\nu(y) \leq M\,\|y\|$ for any $y \in X$ it follows that there is $k \in \mathbb{N}$ such that $\nu(x - x_k) \leq 3$. Let $k \in \mathbb{N}$ be the smallest such number. Then property (H4) implies that $\psi_k(x) \geq \frac{1}{4}$. Thus $\sup_{n \in \mathbb{N}} \psi_n(x) \geq Q$ for each $x \in X$, where $Q = \frac{1}{4}$.

By the continuity of ν there is $\rho > 0$ such that $|\nu(z)| < \frac{1}{2}$ whenever $z \in \tilde{X}$, $\|z\| < \rho$. Now fix $x \in X$ and find an index $j \in \mathbb{N}$ such that $\|x_j - x\| < \rho$. Using the equicontinuity of $\{y \mapsto \operatorname{Im} \nu(w + iy)\}$ at 0 choose $0 < \Delta_j < \Delta$ such that $|\operatorname{Im} \nu(w + iy)| < \delta_j$ whenever $w, y \in X$, $\|y\| < \Delta_j$. Let us define $U_x = \{z = w + iy \in \tilde{X};\ \|z - x_j\| < \rho, \|y\| < \Delta_j\}$. Notice that U_x is an open neighbourhood of x and $z - x_l \in \Omega$ for every $z \in U_x$, $l \in \mathbb{N}$. Let $z = w + iy \in U_x$. Then $|\operatorname{Im} \nu(z - x_l)| = |\operatorname{Im} \nu(w - x_l + iy)| < \delta_j$ for every $l \in \mathbb{N}$. Furthermore, $|\operatorname{Re} \nu(z - x_j)| \leq |\nu(z - x_j)| < \frac{1}{2}$. Hence, by (H3), $|\psi_n(z)| \leq 2^{-n}$ for $n > j$ and $z \in U_x$. It follows that for any $(a_n) \in \ell_\infty$, $(a_n \psi_n(z))_{n \in \mathbb{N}} = \sum_{n=1}^{\infty} a_n \psi_n(z) e_n \in \tilde{c_0}$ and the sum converges absolutely uniformly on U_x. As the mappings $z \mapsto a_n \psi_n(z) e_n$ are holomorphic as mappings from Ω into $\tilde{c_0}$, we can conclude that $(a_n \psi_n)$ is a holomorphic mapping from $G = \bigcup_{x \in X} U_x$ into $\tilde{c_0}$, which gives (P2).

Property (P3) obviously holds by (H2). Finally we show that (P4) is satisfied with $r = 6$. Indeed, fix $n \in \mathbb{N}$. For $x \in X$, $\|x - x_n\| \geq 6$ we have $\nu(x - x_n) \geq \|x - x_n\| - 1 \geq 5$, hence, by (H5), $\psi_n(x) \leq \frac{1}{32} = \frac{Q}{8}$.

□

For the proof of Proposition 5 we need the following version of the Implicit Function Theorem with explicit estimates on the size of the region where the solution is found.

Theorem 9 (Implicit Function Theorem)**.** *Let X be a complex Banach space, $U \subset X$ and $V \subset \mathbb{C}$ open sets, and $F \in H(U \times V)$. Let $\hat{z} \in U$, $\hat{u} \in V$ satisfy $F(\hat{z}, \hat{u}) = 0$. Further let $R > 0$, $S > 0$, and $M > 0$ be such that $B(\hat{z}, S) \subset U$, $B(\hat{u}, R) \subset V$, and $|F(z, u)| \leq M$ for every $z \in B(\hat{z}, S)$, $u \in B(\hat{u}, R)$. Assume that $\left|\frac{\partial F}{\partial u}(\hat{z}, \hat{u})\right| \geq a > 0$ and $0 < r < \frac{aR^2}{aR + M}$. Put $c = ar - \frac{Mr^2}{R(R - r)}$ and $s = S\frac{c}{c + M}$. Then for each $z \in U(\hat{z}, s)$ there is a unique $u \in U(\hat{u}, r)$ satisfying $F(z, u) = 0$. Denote such u by $\varphi(z)$. Then $\varphi \in H(U(\hat{z}, s))$.*

The proof of this theorem is fairly standard using for example the Rouché theorem and Cauchy's estimates for $\frac{\partial^n F}{\partial u^n}$ on $\mathbb{C}$ and for $\frac{\partial F}{\partial z}$ on X. Some details can be found e.g. in [CHP], although the estimates and the proof given there are not entirely correct.

Proof of Proposition 5. Define μ on c_0 as the Minkowski functional of the set $\left\{x \in c_0;\ \sum_{n=1}^{\infty} (x_n)^{2n} \leq 1\right\}$. Then μ is an equivalent norm on c_0 for which $\|x\| \leq \mu(x) \leq \sqrt{2}\,\|x\|$ (see [FPWZ]). This gives property (M4) and (M2).

Let $F\colon \tilde{c}_0 \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$ be defined as $F(z, u) = \sum_{n=1}^{\infty} (z_n/u)^{2n} - 1$. This function is holomorphic on $\tilde{c}_0 \times (\mathbb{C} \setminus \{0\})$ and for every $x \in c_0 \setminus \{0\}$ we have $F(x, \mu(x)) = 0$.

Fix $w \in c_0 \setminus \{0\}$. Put $R = \frac{\|w\|}{2}$, $S = \frac{\|w\|}{4}$, $M = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{2q}{\|w\|}\,|w_n|\right)^{2n}$, $a = \frac{1}{\sqrt{2}q\|w\|}$, $r = \min\left\{\frac{1}{2}\frac{aR^2}{aR + M}, 2 - \sqrt{2}\right\}$, and $\Delta_w = s$ as defined in Theorem 9. Now choose any $y \in c_0$, $|w| \leq |y| \leq q\,|w|$. Then $R < \|w\| \leq \|y\| \leq \mu(y)$, thus $B(\mu(y), R) \subset V = \mathbb{C} \setminus \{0\}$. For any $z \in B(y, S)$, $u \in B(\mu(y), R)$ we have $|u| \geq \mu(y) - R \geq \|y\| - R \geq \|w\| - R = \frac{\|w\|}{2}$ and $|z_n| \leq |y_n| + |z_n - y_n| \leq q\,|w_n| + \|z - y\| \leq q\,|w_n| + \frac{\|w\|}{4}$, and hence $|F(z, u)| \leq 1 + \sum_{n=1}^{\infty} \left|\frac{z_n}{u}\right|^{2n} \leq M$. Finally, $\left|\frac{\partial F}{\partial u}(y, \mu(y))\right| = \left|-\frac{1}{\mu(y)} \sum_{n=1}^{\infty} 2n\left(\frac{y_n}{\mu(y)}\right)^{2n}\right| \geq \frac{1}{\mu(y)} \sum_{n=1}^{\infty} \left(\frac{y_n}{\mu(y)}\right)^{2n} = \frac{1}{\mu(y)} \geq \frac{1}{\sqrt{2}\|y\|} \geq a$. Thus by Theorem 9 the equation $F(z, u) = 0$ uniquely determines a holomorphic function μ_y^w on $U_{\tilde{c}_0}(y, \Delta_w)$ with values in $U(\mu(y), r)$ and this holds for every $y \in c_0$, $|w| \leq |y| \leq q\,|w|$.

Take any two functions $\mu_1 = \mu_{y_1}^{w_1}$, $\mu_2 = \mu_{y_2}^{w_2}$ defined on open balls U_1 and U_2 respectively. If U_1 and U_2 intersect, then it is easy to check that $U_1 \cap U_2 \cap c_0$ is a non-empty set relatively open in c_0. Since $\mu_1 = \mu$ on $U_1 \cap c_0$ and $\mu_2 = \mu$ on $U_2 \cap c_0$, it follows that both holomorphic functions μ_1 and μ_2 agree on some ball in $U_1 \cap U_2$ and therefore on the whole $U_1 \cap U_2$. This observation allows us to put $W = \bigcup \left\{U_{\tilde{c}_0}(y, \Delta_w);\ w \in c_0 \setminus \{0\}, y \in c_0, |w| \le |y| \le q\,|w|\right\}$ and define μ on W by $\mu(z) = \mu_y^w(z)$ whenever $z \in U(y, \Delta_w)$. This gives property (M1).

To prove (M3) let $w \in c_0 \setminus \{0\}$, $y \in c_0$, $|w| \le |y| \le q\,|w|$, $\|y\| \le 1$, and $z \in U_{\tilde{c}_0}(y, \Delta_w)$. Then by the choice of r above we have $\mu(z) \in U(\mu(y), 2-\sqrt{2})$ and therefore $|\mu(z)| < |\mu(y)| + 2 - \sqrt{2} \le \sqrt{2}\,\|y\| + 2 - \sqrt{2} \le 2$.

□

It remains to prove Lemma 3 and Lemma 8. The proofs are standard using integral convolution technique and estimates which are not difficult. We could just write the formulas for the functions in consideration and stop there (we claim a short proof after all). Nevertheless for the convenience of the reader we include rather detailed computations.

Proof of Lemma 8. Let $\zeta_n\colon \mathbb{R}^n \to [0,1]$ be a 1-Lipschitz function (with respect to the maximum norm) such that

$$\zeta_n(x) = \begin{cases} 0 & \text{whenever } x_n \ge 4 \text{ or } \exists i \in \{1,\dots,n-1\}\colon x_i \le 1, \\ 1 & \text{whenever } x_n \le 3 \text{ and } \forall i \in \{1,\dots,n-1\}\colon x_i \ge 2. \end{cases}$$

For each $n \in \mathbb{N}$ put $\delta_n = \sqrt{2^{-n}/8}$ and find $a_n \in \mathbb{R}$ such that

$$a_n 2^{-n} \ge 3, \tag{5}$$

$$e^{-a_n 2^{-n}/8} \le 2\sqrt{\pi}\cdot 2^{-n}, \text{ and} \tag{6}$$

$$\int_{-\sqrt{a_n 2^{-n}}}^{+\infty} e^{-t^2}\,\mathrm{d}t \ge \frac{1}{\sqrt[n]{2}}\sqrt{\pi}. \tag{7}$$

Finally, put

$$\phi_n(z) = \frac{1}{c_n}\int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum\limits_{i=1}^{n} 2^{-i}(z_i - t_i)^2}\,\mathrm{d}t \quad \text{for } z \in \mathbb{C}^n,$$

where $c_n = \int_{\mathbb{R}^n} e^{-a_n \sum_{i=1}^n 2^{-i} t_i^2}\,\mathrm{d}t = \sqrt{\left(\frac{\pi}{a_n}\right)^n \prod_{i=1}^n 2^i}$.

Using standard theorems on integrals dependent on parameter we obtain $\phi_n \in H(\mathbb{C}^n)$. Property (H1) is obvious, and property (H2) is easy to check.

Next we will need the elementary estimate

$$\int_x^{+\infty} e^{-t^2}\,\mathrm{d}t \le \int_x^{+\infty} t e^{-t^2}\,\mathrm{d}t = \frac{1}{2}e^{-x^2} \text{ for } x \ge 1. \tag{8}$$

To prove (H3) we use successively the definition of ζ_n, Fubini's theorem, substitution, $\operatorname{Re} z_j \le \frac{1}{2}$, estimate (8) together with (5), the definition of δ_j, and finally (6) to obtain

$$\begin{aligned}
|\phi_n(z)| &\le \frac{1}{c_n}\int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum\limits_{i=1}^{n} 2^{-i}\operatorname{Re}(z_i - t_i)^2}\,\mathrm{d}t = \frac{e^{a_n \sum\limits_{i=1}^{n} 2^{-i}(\operatorname{Im} z_i)^2}}{c_n}\int_{\mathbb{R}^n} \zeta_n(t) e^{-a_n \sum\limits_{i=1}^{n} 2^{-i}(\operatorname{Re} z_i - t_i)^2}\,\mathrm{d}t \\
&\le \frac{e^{a_n \delta_j^2}}{c_n}\int_{\substack{t \in \mathbb{R}^n \\ t_j > 1}} e^{-a_n \sum\limits_{i=1}^{n} 2^{-i}(\operatorname{Re} z_i - t_i)^2}\,\mathrm{d}t = \frac{e^{a_n \delta_j^2}}{c_n}\int_{\mathbb{R}^{n-1}} e^{-a_n \sum\limits_{i \ne j} 2^{-i}(\operatorname{Re} z_i - t_i)^2}\,\mathrm{d}t \cdot \int_1^{+\infty} e^{-a_n 2^{-j}(\operatorname{Re} z_j - t_j)^2}\,\mathrm{d}t_j \\
&= \frac{e^{a_n \delta_j^2}}{\sqrt{\pi}}\sqrt{a_n 2^{-j}}\int_1^{+\infty} e^{-a_n 2^{-j}(\operatorname{Re} z_j - t)^2}\,\mathrm{d}t = \frac{e^{a_n \delta_j^2}}{\sqrt{\pi}}\int_{\sqrt{a_n 2^{-j}}(1 - \operatorname{Re} z_j)}^{+\infty} e^{-t^2}\,\mathrm{d}t \le \frac{e^{a_n \delta_j^2}}{\sqrt{\pi}}\int_{\frac{1}{2}\sqrt{a_n 2^{-j}}}^{+\infty} e^{-t^2}\,\mathrm{d}t \\
&\le \frac{e^{a_n \delta_j^2}}{2\sqrt{\pi}}\cdot e^{-\frac{1}{4}a_n 2^{-j}} = \frac{1}{2\sqrt{\pi}}\cdot e^{-a_n(2^{-j}/4 - \delta_j^2)} < \frac{1}{2\sqrt{\pi}}\cdot e^{-a_n 2^{-n}/8} \le 2^{-n}.
\end{aligned}$$

To prove (H4) we use successively the definition of ζ_n, Fubini's theorem and substitution, $x_n \le 3$ and $x_i \ge 3$, substitution, and (7) to obtain

$$\begin{aligned}
\phi_n(x) &\ge \frac{1}{c_n}\int_{\substack{t_n \le 3 \\ t_i \ge 2, i=1,\dots,n-1}} e^{-a_n \sum\limits_{i=1}^{n} 2^{-i}(x_i - t_i)^2}\,\mathrm{d}t = \frac{1}{c_n}\int_{-\infty}^{3 - x_n} e^{-a_n 2^{-n} t^2}\,\mathrm{d}t \cdot \prod_{i=1}^{n-1}\int_{2 - x_i}^{+\infty} e^{-a_n 2^{-i} t^2}\,\mathrm{d}t \\
&\ge \frac{1}{c_n}\int_{-\infty}^{0} e^{-a_n 2^{-n} t^2}\,\mathrm{d}t \cdot \prod_{i=1}^{n-1}\int_{-1}^{+\infty} e^{-a_n 2^{-i} t^2}\,\mathrm{d}t \ge \frac{1}{2}\frac{1}{c_n}\prod_{i=1}^{n}\int_{-1}^{+\infty} e^{-a_n 2^{-i} t^2}\,\mathrm{d}t = \frac{1}{2}\frac{1}{(\sqrt{\pi})^n}\prod_{i=1}^{n}\int_{-\sqrt{a_n 2^{-i}}}^{+\infty} e^{-t^2}\,\mathrm{d}t \\
&\ge \frac{1}{2}\frac{1}{(\sqrt{\pi})^n}\prod_{i=1}^{n}\int_{-\sqrt{a_n 2^{-n}}}^{+\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2}\left(\frac{1}{\sqrt{\pi}}\int_{-\sqrt{a_n 2^{-n}}}^{+\infty} e^{-t^2}\,\mathrm{d}t\right)^n \ge \frac{1}{4}.
\end{aligned}$$

Finally, to prove (H5) we use successively the definition of ζ_n, Fubini's theorem, substitution, $x_n \geq 5$, and (8) together with (5) to obtain

$$\begin{aligned}\phi_n(x) &\leq \frac{1}{c_n}\int_{\substack{t\in\mathbb{R}^n\\ t_n<4}} e^{-a_n\sum_{i=1}^{n}2^{-i}(x_i-t_i)^2}\,\mathrm{d}t = \sqrt{\frac{a_n}{\pi 2^n}}\int_{-\infty}^{4} e^{-a_n 2^{-n}(x_n-t)^2}\,\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{a_n 2^{-n}}(4-x_n)} e^{-t^2}\,\mathrm{d}t\\ &\leq \frac{1}{\sqrt{\pi}}\int_{\sqrt{a_n 2^{-n}}}^{+\infty} e^{-t^2}\,\mathrm{d}t \leq \frac{1}{2\sqrt{\pi}}\cdot e^{-a_n 2^{-n}} < \frac{1}{32}.\end{aligned}$$

□

Proof of Lemma 3. Let $\zeta\colon \mathbb{R}\to[0,1]$ be defined as $\zeta(t)=0$ for $t\leq\frac{5}{8}$, $\zeta(t)=4t-\frac{5}{2}$ for $t\in\left(\frac{5}{8},\frac{7}{8}\right)$, and $\zeta(t)=1$ for $t\geq\frac{7}{8}$. Obviously ζ is a 4-Lipschitz function. For each $n\in\mathbb{N}$ find $a_n\in\mathbb{R}$ such that

$$\frac{3}{8}\sqrt{a_n}\geq 1, \tag{9}$$

$$e^{-\frac{5}{64}a_n}\leq 2\sqrt{\pi}\cdot 2^{-n}, \tag{10}$$

$$\int_{-\frac{1}{8}\sqrt{a_n}}^{+\infty} e^{-t^2}\,\mathrm{d}t \geq (1-2^{-n})\sqrt{\pi}, \text{ and} \tag{11}$$

$$2\sqrt{a_n}\cdot e^{-\frac{1}{64}a_n}\leq\sqrt{\pi}\cdot 2^{-n}. \tag{12}$$

Finally, put

$$\theta_n(z)=\frac{1}{c_n}\int_{\mathbb{R}}\zeta(t)e^{-a_n(z-t)^2}\,\mathrm{d}t \quad \text{for } z\in\mathbb{C},$$

where $c_n=\int_{\mathbb{R}}e^{-a_n t^2}\,\mathrm{d}t=\sqrt{\frac{\pi}{a_n}}$.

Using standard theorems on integrals dependent on parameter we obtain $\theta_n\in H(\mathbb{C})$. Property (T1) is obvious, and property (T2) is easy to check.

To prove (T3) we use successively the definition of ζ, $|\operatorname{Im} z|\leq\frac{1}{4}$, substitution, $\operatorname{Re} z\leq\frac{1}{4}$, estimate (8) together with (9), and finally (10) to obtain

$$\begin{aligned}|\theta_n(z)| &\leq \frac{1}{c_n}\int_{\mathbb{R}}\zeta(t)e^{-a_n\operatorname{Re}(z-t)^2}\,\mathrm{d}t = \frac{e^{a_n(\operatorname{Im} z)^2}}{c_n}\int_{\mathbb{R}}\zeta(t)e^{-a_n(\operatorname{Re} z-t)^2}\,\mathrm{d}t \leq \frac{e^{\frac{1}{16}a_n}}{c_n}\int_{\frac{5}{8}}^{+\infty}e^{-a_n(\operatorname{Re} z-t)^2}\,\mathrm{d}t\\ &= \frac{e^{\frac{1}{16}a_n}}{\sqrt{\pi}}\int_{\sqrt{a_n}\left(\frac{5}{8}-\operatorname{Re} z\right)}^{+\infty}e^{-t^2}\,\mathrm{d}t \leq \frac{e^{\frac{1}{16}a_n}}{\sqrt{\pi}}\int_{\frac{3}{8}\sqrt{a_n}}^{+\infty}e^{-t^2}\,\mathrm{d}t \leq \frac{e^{\frac{1}{16}a_n}}{2\sqrt{\pi}}\cdot e^{-\frac{9}{64}a_n} = \frac{e^{-\frac{5}{64}a_n}}{2\sqrt{\pi}} \leq 2^{-n}.\end{aligned}$$

To prove (T4) we use successively the definition of ζ, substitution, $x\geq 1$, and (11) to obtain

$$\theta_n(x)\geq\frac{1}{c_n}\int_{\frac{7}{8}}^{+\infty}e^{-a_n(x-t)^2}\,\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_{\sqrt{a_n}\left(\frac{7}{8}-x\right)}^{+\infty}e^{-t^2}\,\mathrm{d}t \geq \frac{1}{\sqrt{\pi}}\int_{-\frac{1}{8}\sqrt{a_n}}^{+\infty}e^{-t^2}\,\mathrm{d}t\geq 1-2^{-n}.$$

Finally, we show (T5). Differentiating under the integral sign we obtain

$$\theta_n'(x)=\frac{2a_n}{c_n}\int_{\mathbb{R}}\zeta(t)(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t.$$

Thus for $x\leq\frac{1}{2}$ using the definition of ζ, substitution, and (12) we get

$$\left|\theta_n'(x)\right|\leq\frac{2a_n}{c_n}\int_{\frac{5}{8}}^{+\infty}(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t = \frac{1}{c_n}\int_{-\infty}^{-a_n\left(\frac{5}{8}-x\right)^2}e^y\,\mathrm{d}y = \sqrt{\frac{a_n}{\pi}}\cdot e^{-a_n\left(\frac{5}{8}-x\right)^2}\leq\sqrt{\frac{a_n}{\pi}}\cdot e^{-\frac{1}{64}a_n}\leq 2^{-n}.$$

On the other hand, for $x\geq 1$ using the definition of ζ, evaluation of the integrals, and (12) we get

$$\begin{aligned}\left|\theta_n'(x)\right| &= \frac{2a_n}{c_n}\left|\int_{\frac{5}{8}}^{\frac{7}{8}}\zeta(t)(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t+\int_{\frac{7}{8}}^{+\infty}(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t\right|\\ &\leq \frac{2a_n}{c_n}\int_{\frac{5}{8}}^{\frac{7}{8}}|t-x|\,e^{-a_n(t-x)^2}\,\mathrm{d}t+\frac{2a_n}{c_n}\left|\int_{\frac{7}{8}}^{+\infty}(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t\right|\\ &= \frac{-2a_n}{c_n}\int_{\frac{5}{8}}^{\frac{7}{8}}(t-x)e^{-a_n(t-x)^2}\,\mathrm{d}t+\frac{1}{c_n}\cdot e^{-a_n\left(\frac{7}{8}-x\right)^2}\\ &= \frac{1}{c_n}\left(e^{-a_n\left(\frac{7}{8}-x\right)^2}-e^{-a_n\left(\frac{5}{8}-x\right)^2}+e^{-a_n\left(\frac{7}{8}-x\right)^2}\right)\leq\frac{2}{c_n}\cdot e^{-a_n\left(\frac{7}{8}-x\right)^2}\leq 2\sqrt{\frac{a_n}{\pi}}\cdot e^{-\frac{1}{64}a_n}\leq 2^{-n}.\end{aligned}$$

□

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