LOCALLY FLAT BANACH SPACES

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ABSTRACT. Let X be a separable Banach space with a Schauder basis, admitting a bump which depends locally on finitely many coordinates. Then X admits also a C^∞-smooth bump which depends locally on finitely many coordinates. There exists an Orlicz space admitting a C^∞-smooth bump which depends locally on finitely many coordinates, and which is not isomorphic to a subspace of any $C(K)$, K scattered. In view of the related results this space is possibly not isomorphic to a polyhedral space.

1. INTRODUCTION

In the present paper we investigate the properties of separable Banach spaces admitting bump functions depending locally on finitely many coordinates (LFC). The first use of the LFC notion for a function was the construction of C^∞-smooth and LFC renorming of c_0, due to Kuiper, which appeared in [BF]. The LFC notion was explicitly introduced and investigated in the paper [PWZ] of Pechanec, Whitfield and Zizler. In their work the authors have proved that every Banach space admitting a LFC bump is saturated with copies of c_0, providing in some sense a converse to Kuiper's result. Not surprisingly, it turns out that the LFC notion is closely related to the class of polyhedral spaces, introduced by Klee [K] and thoroughly investigated by many authors (see [JL, Chapter 15] for results and references). (We note that polyhedrality is understood in the isomorphic sense in this paper.) Indeed, prior to [PWZ], Fonf [F1] has proved that every polyhedral Banach space is saturated with copies of c_0. Later, it was independently proved in [F2] and [Haj1] that every separable polyhedral Banach space admits an equivalent LFC norm. Using the last result Fonf's result is a corollary of [PWZ]. The notion of LFC has been exploited (at least implicitly) in a number of papers, in order to obtain very smooth bump functions, norms and partitions of unity on non-separable Banach spaces, see e.g. [To], [Ta], [DGZ1], [GPWZ], [GTWZ], [FZ], [Hay1], [Hay2], [Hay3], [S1], [S2], [Haj1], [Haj2], [Haj3], and the book [DGZ]. In fact, it seems to be the only general approach to these problems. The reason is simple; it is relatively easy to check the (higher) differentiability properties of functions of several variables, while for functions defined on a Banach space it is very hard.

For separable spaces, one of the main known results is that a separable Banach space is polyhedral if and only if it admits a LFC renorming (resp. C^∞-smooth and LFC renorming), [Haj1]. However, this smoothing up result is obtained by using the boundary of a Banach space, rather than through some direct smoothing procedure. There is a variety of open questions, well known among the workers in the area, concerning the existence and possible smoothing of general non-convex LFC functions. In our note we are going to address the following ones. Suppose a Banach space X admits a LFC bump. Does X admit a C^∞-smooth bump (norm)? Is the space X necessarily polyhedral?

To this end, we develop some basic theory of LFC functions on separable Banach spaces. In fact, in Section 2 we introduce a formally more general notion of a locally flat space, and generalise the known structural results valid for spaces admitting a (continuous) LFC bump function in this context. It is not clear to us whether the generalisation is genuine. However, locally flat spaces include for example all spaces admitting a (not necessarily continuous) bump locally depending on finitely many linear (i.e. not necessarily continuous) functionals. This notion offers itself for a possible purely combinatorial characterisation of locally flat spaces. We intend to investigate in this direction in the future.

The main result of Section 3 is that a separable Banach space with a Schauder basis has a C^∞-smooth and LFC bump function whenever it has a continuous LFC bump. This seems to be the first relatively general result in this direction. We establish some additional properties of such bumps, with an eye on the future developments.

The main result of the paper, contained in Section 5, is a certain rather subtle construction of an Orlicz sequence space having a C^∞-smooth and LFC bump function, which we suspect to be non-polyhedral. Such an example is of course needed to justify the whole theory, since in the polyhedral case the smoothing up (and structural) results are well known and easier. In fact, our paper, and in particular the example was motivated by the beautiful theory of polyhedrality for separable Banach spaces with Schauder basis, and especially Orlicz sequence spaces, developed by Leung in [L1] and [L2]. The key result of these works is the following theorem.

Theorem ([L2]). *The following statements are equivalent for every non-degenerate Orlicz function M:*

(i) There exists a constant $K > 0$ such that $\lim\limits_{t\to 0+} \frac{M(Kt)}{M(t)} = \infty$.

(ii) The Orlicz sequence space h_M is isomorphic to a subspace of $C(\omega^\omega)$.

(iii) The Orlicz sequence space h_M is isomorphic to a subspace of $C(K)$ for some scattered compact K.

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All spaces satisfying (ii) are polyhedral, and Leung conjectured that conversely all polyhedral Orlicz sequence spaces fall under this description. There is a strong evidence supporting this idea. First, Theorem 34, part of which is also in Leung's paper, shows that the naturally defined LFC renormings exist precisely for those spaces. Second, negating the condition in (i) we obtain the following formula

$$(\forall K > 0)(\exists \{t_n\}_{n=1}^{\infty}, t_n \searrow 0) \lim_{n\to\infty} \frac{M(Kt_n)}{M(t_n)} < \infty.$$

Reversing the order of the quantifiers we obtain the following stronger (less general) condition

$$(\exists \{t_n\}_{n=1}^{\infty}, t_n \searrow 0)(\forall K > 0) \lim_{n\to\infty} \frac{M(Kt_n)}{M(t_n)} < \infty.$$

Leung proved that Orlicz sequence spaces satisfying the last condition are not polyhedral (although they may be c_0 saturated).

Thus Leung's theorem above is a near characterisation of polyhedrality for Orlicz sequence spaces, the gap lying in the exchange of quantifiers. Our example of an Orlicz sequence space with C^{∞}-smooth and LFC bump lies strictly in between the above conditions. Therefore, our space is either a non-polyhedral space admitting a LFC bump (we are inclined to believe this alternative), or Leung's polyhedral conjecture is false.

We use a standard Banach space notation. If $\{e_i\}$ is a Schauder basis of a Banach space, we denote by $\{e_i^*\}$ its biorthogonal functionals. P_n are the canonical projections associated with the basis $\{e_i\}$, P_n^* are the operators adjoint to P_n, i.e. the canonical projections associated with the basis $\{e_i^*\}$. Given a set $A \subset \mathbb{N}$ we denote by P_A the projection associated with the set A, i.e. $P_A x = \sum_{i\in A} e_i^*(x)e_i$. By R_n we denote the projections $R_n = I - P_n$, where I is the identity operator. For a finite set B, $|B|$ denotes the number of elements of B. $U(x,\delta)$ is an open ball centred at x with radius δ. By $X^{\#}$ we denote an algebraic dual to a vector space X.

We refer to [F–Z], [LT] and [JL] for background material and results.

2. Local Flatness

In this section, we are going to generalise the well known structural results for polyhedral spaces (or spaces with a (continuous) LFC bump), to a (at least formally) larger class of locally flat spaces. Apart from the usual effort to find the essential ingredients in the theory, we feel that the more discrete and combinatorial notions have a better chance for finding characterisation, e.g. among the Orlicz sequence spaces. This is crucial for finding new examples.

The notion of a function, defined on a Banach space with a Schauder basis, which is locally dependent on finitely many coordinates was introduced in [PWZ]. The following definition is a slight generalisation which was used by many authors.

Definition 1. *Let X be a topological vector space, $\Omega \subset X$ an open subset, E be an arbitrary set, $M \subset X^{\#}$ and $g\colon \Omega \to E$. We say that g* depends only on M *on a set $U \subset \Omega$ if $g(x) = g(y)$ whenever $x, y \in U$ are such that $f(x) = f(y)$ for all $f \in M$. We say that g* depends locally on finitely many coordinates from M *(LFC-M for short) if for each $x \in \Omega$ there are a neighbourhood $U \subset \Omega$ of x and a finite subset $F \subset M$ such that g depends only on F on U. We say that g* depends locally on finitely many coordinates *(LFC for short) if it is LFC-X^*.*

We may equivalently say that g depends only on $\{f_1,\dots,f_n\} \subset X^{\#}$ on $U \subset \Omega$ if there exist a mapping $G\colon \mathbb{R}^n \to E$ such that $g(x) = G(f_1(x),\dots,f_n(x))$ for all $x \in U$. If g is moreover LFC (i.e. LFC-X^*), then we have the following characterisation:

Lemma 2. *Let X be a topological vector space, $\Omega \subset X$ an open subset, E be an arbitrary set, $M \subset X^*$ and $g\colon \Omega \to E$. The mapping g is LFC-M if and only if for each $x \in \Omega$ there are an open neighbourhood $U \subset \Omega$ of x, $n \in \mathbb{N}$, a biorthogonal system $\{(e_i, f_i)\}_{i=1}^n \subset X \times M$, an open set $V \subset \mathbb{R}^n$, and a mapping $G\colon V \to E$, such that $g(y) = G\big(f_1(y),\dots,f_n(y)\big)$ for all $y \in U$, where $G(w) = g\left(x + \sum_{i=1}^n \big(w_i - f_i(x)\big)e_i\right)$ for each $w \in V$.*

Proof. Let U_0 be an open neighbourhood of x and $n \in \mathbb{N}$ such that g depends only on $\{f_1,\dots,f_n\} \subset M$ on U_0. Without loss of generality we may assume that the functionals $f_1,\dots,f_n$ are linearly independent. Hence there are vectors $e_1,\dots,e_n \in X$ such that $\{(e_i, f_i)\}_{i=1}^n$ is a biorthogonal system (as $\bigcap_{i\neq j} \ker f_i \not\subset \ker f_j$ for each $1 \le j \le n$). Let $\Phi\colon \mathbb{R}^n \to X$ be defined as $\Phi(w) = x + \sum_{i=1}^n \big(w_i - f_i(x)\big)e_i$. This is a continuous mapping, so the set $V = \Phi^{-1}(U_0)$ is an open subset of $\mathbb{R}^n$. Notice that $G(w) = g(\Phi(w))$ for each $w \in V$. Let $\Psi\colon X \to \mathbb{R}^n$ be defined as $\Psi(y) = (f_1(y),\dots,f_n(y))$. This is a continuous mapping, so the set $U = \Psi^{-1}(V) \cap U_0$ is open. Moreover, $\Phi(\Psi(x)) = x$, hence U is an open neighbourhood of x.

Now choose any $y \in U$. Since $\Psi(y) \in V$, $G(\Psi(y))$ is well defined. Further, from the facts that $y \in U_0$, $\Phi(\Psi(y)) \in U_0$, $f_j(\Phi(\Psi(y))) = f_j(x) + \sum_{i=1}^n \big(f_i(y) - f_i(x)\big) f_j(e_i) = f_j(y)$ for each $1 \le j \le n$, and g depends only on $\{f_1,\dots,f_n\}$ on U_0, we may conclude that $G(\Psi(y)) = g(\Phi(\Psi(y))) = g(y)$.

The other implication is obvious. $\square$

Notice, that if $g\colon \Omega \to E$ is LFC and $h\colon E \to F$ is any mapping, then also $h \circ g$ is LFC.

The canonical example of a non-trivial LFC function is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin. Indeed, take any $x = (x_i) \in c_0$, $x \neq 0$. Let $n \in \mathbb{N}$ be such that $|x_i| < \|x\|_{\infty}/2$ for $i > n$. Then $\|\cdot\|_{\infty}$ depends only on $\{e_1^*,\dots,e_n^*\}$ on $U(x, \|x\|_{\infty}/4)$.

The following lemma shows that under some conditions it is possible to join together some of the neighbourhoods in the definition of LFC:

Lemma 3. *Let X be a topological vector space, E be an arbitrary set, $g\colon X \to E$ and $M \subset X^{\#}$. Let $U_\alpha \subset X$, $\alpha \in I$ be open sets such that $U = \bigcup_{\alpha\in I} U_\alpha$ is convex and g depends only on M on each U_α, $\alpha \in I$. Then g depends only on M on the whole of U.*

Proof. Pick any $x, y \in U$ such that $f(x) = f(y)$ for all $f \in M$. Since U is convex, the line segment $[x, y] \subset U$. Since $[x, y]$ is compact, there is a finite covering $U_1, \dots, U_n \in \{U_\alpha\}_{\alpha\in I}$ of $[x, y]$. Since $[x, y]$ is connected, without loss of generality we may assume that $x \in U_1$, $y \in U_n$ and there are $x_i \in U_i \cap U_{i+1} \cap [x, y]$ for $i = 1, \dots, n-1$. As $x_i \in [x, y]$, we have $f(x) = f(y) = f(x_i)$ for all $f \in M$ and $i = 1, \dots, n-1$. Therefore $g(x) = g(x_1) = \dots = g(x_{n-1}) = g(y)$. □

A norm on a normed space is said to be LFC, if it is LFC away from the origin. Recall that a bump function (or bump) on a topological vector space X is a function $b\colon X \to \mathbb{R}$ with a bounded non-empty support.

The existence of a LFC norm (or even a continuous LFC bump) on a Banach space is known to have strong implications on the structure of the space (see e.g. [F1], [PWZ], [FZ]). The role of continuity in these results seems rather interesting. It turns out that the essence lies in the discrete (or combinatorial) structure of the space itself. This leads us to the following general concept:

Definition 4. *Let X be a vector space, $A \subset X$, $U \subset X$ be arbitrary subsets of X. We say that A is* determined *on U by a subspace $Z \subset X$ if $U \cap (y + Z) \subset A$ for all $y \in U \cap A$.*

Clearly, if A is determined on U by Z then A is determined on U by any subspace of Z.

Let us denote the set of all finite-codimensional subspaces of a vector space X by $\mathcal{FC}(X)$. If X is moreover a topological vector space, we denote by $\mathcal{FC}_c(X)$ the set of all closed finite-codimensional subspaces.

Definition 5. *Let X be a topological vector space, $A \subset X$ be an arbitrary subset of X and $\mathcal{Z} \subset \mathcal{FC}(X)$. We say that A is* locally finite-dimensionally determined *by $\mathcal{Z}$ (or LFD-$\mathcal{Z}$ for short) if for any $x \in X$ there is a neighbourhood $U \subset X$ of x and $Z \in \mathcal{Z}$ such that A is determined by Z on U. We say that A is* locally finite-dimensionally determined *(or LFD) if A is LFD-$\mathcal{FC}(X)$.*

Fact 6. *Let X be a topological vector space, let $A \subset X$ and $M \subset X^{\#}$. The function χ_A is LFC-M if and only if A is LFD-$\mathcal{Z}$ for $\mathcal{Z} = \{\bigcap_{i=1}^n \ker f_i;\ \{f_1, \dots, f_n\} \subset M, n \in \mathbb{N}\}$.*

Proof. A is determined on U by $\bigcap_{i=1}^n \ker f_i$ if and only if χ_A on U depends only on $\{f_1, \dots, f_n\} \subset X^{\#}$. □

Fact 7. *Let X be a topological vector space and $A, B \subset X$.*

(a) X and $\emptyset$ are LFD. If A and B are LFD, then so are the sets $A \cap B$, $A \cup B$ and $X \setminus A$. In other words, all LFD subsets of X form an algebra.
(b) If $T : X \to X$ is an automorphism or a translation and A is LFD, then $T(A)$ is also LFD.
(c) If A and B are separated (i.e. $A \cap \overline{B} = \overline{A} \cap B = \emptyset$) and $A \cup B$ is LFD, then both A and B are LFD.

Proof. (a): Fix $x \in X$. If U, V are neighbourhoods of x such that A is determined by Z on U and B is determined by W on V, then both $A \cap B$ and $A \cup B$ are determined by $Z \cap W$ on $U \cap V$. The rest is obvious.

(b): It is obvious, since an automorphism preserves the finite codimension of subspaces.

(c): For a fixed $x \in X$ there is a neighbourhood of x such that $A \cup B$ is determined by $Z \in \mathcal{FC}(X)$ on U and $U - x$ is balanced, hence U is connected. For any $y \in U \cap A$, we have $U \cap (y + Z) \subset A \cup B$ and $U \cap (y + Z)$ is connected, which means that $U \cap (y + Z) \subset A$. □

Theorem 8. *Let X be a topological vector space and $\mathcal{Z} \subset \mathcal{FC}(X)$. If a set $A \subset X$ is LFD-$\mathcal{Z}$, then its closure $\overline{A}$ is LFD-$\widetilde{\mathcal{Z}}$, where $\widetilde{\mathcal{Z}} = \{\overline{Z},\ Z \in \mathcal{Z}\} \subset \mathcal{FC}_c(X)$.*

Proof. Fix $x \in X$. There is an open neighbourhood of zero U and $Z \in \mathcal{Z}$ such that A is determined on $x + U$ by Z. Let V be an open neighbourhood of zero such that $V + V + V \subset U$. Choose any $y \in (x + V) \cap \overline{A}$ and $z \in \overline{Z}$ such that $y + z \in x + V$. There is a net $\{y_\gamma\} \subset A$ such that $y_\gamma \to y$ and a net $\{z_\gamma\} \subset Z$ such that $z_\gamma \to z$. We can moreover assume that $\{y_\gamma\} \subset x + U$, $\{y_\gamma\} \subset y + V$ and $\{z_\gamma\} \subset z + V$. Then $y_\gamma + z_\gamma - x = (y + z - x) + (y_\gamma - y) + (z_\gamma - z) \in V + V + V \subset U$. Thus $y_\gamma + z_\gamma \in x + U$ which together with $y_\gamma \in (x + U) \cap A$ gives $y_\gamma + z_\gamma \in A$. It follows that $y + z \in \overline{A}$, which means that $\overline{A}$ is determined on $x + V$ by $\overline{Z}$. □

Similarly, we have

Theorem 9. *Let X be a topological vector space, $\Omega \subset X$ an open subset, E a Hausdorff topological space and $g\colon \Omega \to E$. If g is LFC-$X^{\#}$ and continuous, then g is LFC-X^*.*

Proof. Fix $x \in \Omega$. Let U be a neighbourhood of x such that g depends only on $\{f_1, \dots, f_n\} \subset X^{\#}$ on U. Let $\{\tilde{f}_1, \dots, \tilde{f}_n\} \subset X^*$ are such that $\bigcap \ker \tilde{f}_i = \overline{\bigcap \ker f_i}$. Choose $y \in U$. Since $g(z) = g(y)$ for any $z \in U$ such that $z \in y + \bigcap \ker f_i$, the continuity of g implies that $g(z) = g(y)$ also for any $z \in U$ such that $z \in \overline{y + \bigcap \ker f_i}$, i.e. whenever $\tilde{f}_i(y) = \tilde{f}_i(z)$ for all $1 \le i \le n$. □

If X is a topological vector space, let us recall that a set-valued mapping $\psi\colon X \to 2^X$ is called a cusco mapping if for each $x \in X$, $\psi(x)$ is a non-empty compact convex subset of X and for each open set U in X, $\{x \in X; \psi(x) \subset U\}$ is open.

Lemma 10. *Let X be a locally convex space, E be an arbitrary set and $g\colon X \to E$ be a LFC-M mapping for some $M \subset X^\#$. Further, let $\psi\colon X \to 2^X$ be a cusco mapping with the following property: For any finite $F \subset M$, if $x, y \in X$ are such that $f(x) = f(y)$ for all $f \in F$, then for each $w \in \psi(x)$ there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$.*

Then the mapping $G\colon X \to 2^E$, $G(x) = g(\psi(x))$, is LFC-M.

Proof. Let $x_0 \in X$. We can find a finite covering of the compact $\psi(x_0)$ by open sets U_i, $i = 1, \dots, n$, so that g depends only on a finite set $F_i \subset M$ on U_i. Let W be a convex neighbourhood of zero such that $\psi(x_0) + W \subset \bigcup U_i$ and put $U = \psi(x_0) + W$ and $F = \bigcup F_i$. As U is convex and $U \subset \bigcup U_i$, by Lemma 3, g depends only on F on U.

Suppose $V \subset X$ is a neighbourhood of x_0 such that $\psi(V) \subset U$. Let $x, y \in V$ be such that $f(x) = f(y)$ for all $f \in F$. Choose $w' \in G(x)$ and find $w \in \psi(x)$ for which $g(w) = w'$. Then, by the assumption on ψ, there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$. But we have also $w \in \psi(x) \subset U$ and $z \in \psi(y) \subset U$ and hence $g(w) = g(z)$. Therefore $w' \in G(y)$ and by the symmetry we can conclude that $G(x) = G(y)$. □

As we shall see, the existence of a non-empty bounded LFD set in an infinite-dimensional space has a strong impact on the structure of the space.

Definition 11. *We say that a topological vector space X is* locally flat *if there exists a non-empty* bounded *LFD subset $A \subset X$.*

Let X be a topological vector space, $Y \subset X$ and $Z \subset X$ be linear subspaces. As follows from the remark after Definition 4 and the fact that $\dim Y/(Y \cap Z) \leq \dim X/Z$, any linear subspace of a locally flat space is also locally flat.

By Theorem 8 and Fact 6, X is locally flat if and only if it admits a LFC bump function b (in general arbitrary, i.e. even non-continuous). Indeed, then $(1 - \chi_{\{0\}}) \circ b$ is a characteristic function of a bounded set which is LFC.

Theorem 12. *Let X be a locally flat topological vector space. Then X has a basis of neighbourhoods of zero formed by bounded LFD sets.*

Proof. It suffices to show that there is a set $C \subset X$ that is a bounded LFD neighbourhood of zero in X, since then by the boundedness $\{\frac{1}{n}C\}_{n=1}^\infty$ is a basis of neighbourhoods of zero.

By Fact 7 and Theorem 8 we may assume that there is a closed bounded LFD-$\mathcal{FC}_c(X)$ subset A of X such that $0 \in A$. There is a neighbourhood U of zero and $Y \in \mathcal{FC}_c(X)$ such that A is determined by Y on U. Put $A_0 = A \cap Y$. By Fact 7, A_0 is still a closed bounded LFD-$\mathcal{FC}_c(X)$ subset of X for which $0 \in U \cap Y \subset A_0 \subset Y$.

We assume that $\operatorname{codim} Y = 1$, otherwise we repeat inductively the following construction.

Choose $e \in X \setminus Y$ and denote $B = \{se;\ |s| \leq 1\}$. Put $A_1 = A_0 + B$. The set A_1 is bounded and LFD-$\mathcal{FC}_c(X)$: Fix any $x \in X$, $x = y + te$ for $y \in Y$ and t scalar. There is a neighbourhood V of y such that A_0 is determined on V by some $Z \in \mathcal{FC}_c(X)$, $Z \subset Y$. We denote $V_Y = V \cap Y$ and put $W = V_Y + te + B$. Since Y is closed and $\operatorname{codim} Y = 1$, the product topology on $Y \oplus \operatorname{span}\{e\}$ coincides with the topology of X and thus W is a neighbourhood of x. Then for any $z \in W \cap A_1$ we have $z = z_1 + se$, where $z_1 \in V_Y \cap A_0 = V \cap A_0$ and $|s| \leq 1$. As A_0 is determined by Z on V, we have $V \cap (z_1 + Z) \subset A_0$ and therefore $W \cap (z + Z) = V_Y \cap (z_1 + Z) + se \subset A_0 + se \subset A_1$.

A_1 is a neighbourhood of zero in X, because $A_1 \supset (U \cap Z) + B$ and $U \cap Z$ is a neighbourhood of zero in Z and we use the same argument on product topology as above. □

Using Kolmogorov's theorem we immediately obtain

Corollary 13. *Any Hausdorff locally convex space that is locally flat is normable.*

Another consequence follows from Lemma 10.

Corollary 14. *Let X be a locally flat normed linear space. Then X has a balanced bounded LFD neighbourhood of zero.*

Proof. By Theorem 12 there is $A \subset X$ which is a bounded LFD neighbourhood of zero. Define a mapping $\psi\colon X \to 2^X$ by $\psi(x) = \{tx;\ |t| \leq 1\}$. It is easy to check that ψ is a cusco mapping. Furthermore, let $F \subset X^\#$, and suppose $x, y \in X$ are such that $f(x) = f(y)$ for all $f \in F$. Choose any $w \in \psi(x)$. Then $w = tx$ for some suitable t, $|t| \leq 1$, and we have $ty \in \psi(y)$ and $f(w) = f(tx) = f(ty)$ for all $f \in F$. The function $g = \chi_A$ is LFC by Fact 6. Thus Lemma 10 implies that the function $h(x) = \inf_{|t|\leq 1} g(tx) = \inf g(\psi(x))$ is LFC.

Let $D = h^{-1}(\{1\})$. This set is LFD by Fact 6. We have $h(x) \leq g(x)$ for all $x \in X$ and hence $D \subset A$ and D is bounded. Since A is a neighbourhood of zero, there is some ball B, $B \subset A$, and we have $h(x) = 1$ for any $x \in B$. Thus $B \subset D$ and D is a neighbourhood of zero. Next, $h(tx) = \inf_{|s|\leq 1} g(tsx) \geq \inf_{|s|\leq 1} g(sx) = h(x)$ for any t, $|t| \leq 1$. Therefore $x \in D$ implies $tx \in D$ for all t, $|t| \leq 1$ and D is balanced. □

Theorem 15. *Let X be a normed linear space, $A \subset X$ be a balanced bounded LFD neighbourhood of zero. If the Minkowski functional p of A is continuous, then it is LFC away from the origin. In particular, if A is moreover convex, then p is an equivalent LFC norm.*

Proof. Without loss of generality we may assume that A is closed and LFD-$\mathcal{FC}_c(X)$.

Fix any $x \in X \setminus \{0\}$ and put $\beta = p(x)$. There is $0 < \delta < \|x\|$ such that βA is determined by $Z \in \mathcal{FC}_c(X)$ on $U(x,\delta)$. Let $t_1 = \left(1 + \frac{\|x\|}{\|x\|+\delta}\right)/2$ and $t_2 = \left(1 + \frac{\|x\|}{\|x\|-\delta}\right)/2$. Let V be a neighbourhood of x such that $|p(y) - p(x)| < \beta \min\{1 - t_1, t_2 - 1\}$ for $y \in V$. Put

$$U = V \cap \bigcap_{t_1 < t < t_2} U(tx, t\delta) = V \cap U(t_1 x, t_1\delta) \cap U(t_2 x, t_2\delta),$$

which is a neighbourhood of x, as by the definition of t_1 and t_2 both $U(t_1 x, t_1\delta)$ and $U(t_2 x, t_2\delta)$ are neighbourhoods of x. (The second equality follows by an easy convexity argument.)

It is easy to see that each of the sets $t\beta A$, $t_1 < t < t_2$, is determined on U by Z. Furthermore, $t_1\beta < p(y) < t_2\beta$ for any $y \in U$. Since A is closed, we have $y \in p(y)A$ and $y \notin tA$ for $0 < t < p(y)$. Therefore $y + z \in p(y)A$ and $y + z \notin tA$ for $t_1\beta < t < p(y)$ whenever $z \in Z$ is such that $y + z \in U$. As A is balanced, it follows that $y + z \notin tA$ for all $0 < t < p(y)$ and hence $p(y + z) = p(y)$ whenever $z \in Z$ is such that $y + z \in U$. This means that p depends on U only on $f_1, \dots, f_n \in X^*$ such that $Z = \bigcap \ker f_i$.

□

Theorem 16 ([PWZ]). *An infinite-dimensional locally flat Banach space X is saturated by c_0.*

Proof. As any subspace of X is also locally flat, it suffices to prove that $c_0 \subset X$.

Let $A \subset X$ be a non-empty bounded LFD set. Without loss of generality we may assume that $0 \in A$. We will inductively construct a sequence $\{x_i\} \subset X$ satisfying $\sum_{i=0}^{n} \varepsilon_i x_i \in A$ for all choices of signs $\varepsilon_i = \pm 1$, $i = 0, \dots, n$, as follows: Set $x_0 = 0$. If $x_0, x_1, \dots, x_{n-1}$ have already been defined, we put

$$A_n = \left\{ y \in X \setminus \{0\};\ \sum_{i=0}^{n-1} \varepsilon_i x_i + \varepsilon_n y \in A \quad \text{for all choices of signs } \varepsilon_i = \pm 1, i = 0, \dots, n \right\}.$$

Since A is LFD, the set A_n is non-empty. Indeed, $\sum_{i=0}^{n-1} \varepsilon_i x_i \in A$ for any $\varepsilon_i = \pm 1$ by the induction and in the neighbourhood of each of these points the set A is determined by some finite-codimensional subspace. Since there is finitely many of these points, the intersection of all the respective finite-codimensional subspaces is non-empty and sufficiently small vectors from this intersection belong to A_n. We put $M_n = \sup_{y \in A_n} \|y\|$, and choose $x_n \in A_n$ such that $\|x_n\| > M_n/2$.

We claim that the series $\sum_{i=1}^{\infty} x_i$ does not converge unconditionally. Indeed, let us assume the contrary. Then the set $S = \left\{ \sum_{i=1}^{n} \varepsilon_i x_i;\ \varepsilon_i = \pm 1, n \in \mathbb{N} \right\} \subset A$ is relatively compact and we can find a finite covering of the compact $\overline{S}$ by open balls $U(a_1, \delta_1), \dots, U(a_k, \delta_k)$ and $Z_1, \dots, Z_k \subset \mathcal{FC}(X)$ such that A is determined on $U(a_i, 2\delta_i)$ by Z_i, $i = 1, \dots, k$. We put $Z = \bigcap_{i=1}^{k} Z_i$ and $\delta = \min_{1 \le i \le k} \delta_i$. As $\dim Z = \infty$ (and hence Z is non-trivial), we can choose $z \in Z$ for which $\|z\| = \delta$. Since $z \in A_n$ for any $n \in \mathbb{N}$, it follows that $\|x_n\| > M_n/2 \ge \delta/2$ for all $n \in \mathbb{N}$, which contradicts the convergence of $\sum_{i=1}^{\infty} x_i$.

Without loss of generality we may assume that $\sum_{i=1}^{\infty} x_i$ is not convergent (otherwise we change appropriately the signs of x_i). As the set A is bounded, there is $K > 0$ such that $\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \le K$ for any choice of $\varepsilon_i = \pm 1$ and all $n \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} x_i$ is weakly unconditionally Cauchy and by the Bessaga-Pełczyński theorem ([LT, 2.e.4]) X contains an isomorphic copy of c_0. (The canonical basis of c_0 is equivalent to some sequence of blocks of $\{x_i\}$.)

□

Theorem 17. *Let $A \subset X$ be a non-empty bounded LFD-$\mathcal{Z}$ subset of a Banach space X. Denote $\mathcal{Z}^{\perp} = \bigcup\{Z^{\perp};\ Z \in \mathcal{Z}\}$. Then $\overline{\mathcal{Z}^{\perp}} = X^*$.*

Proof. Since $Z^{\perp} = \overline{Z}^{\perp}$, by Theorem 8 we may assume that A is closed. Pick any $f \in X^*$ and $\varepsilon > 0$ and notice that f is bounded on A. Let $\mathfrak{I}_A \colon X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of the set A, i.e. $\mathfrak{I}_A(x) = 0$ for $x \in A$ and $\mathfrak{I}_A = +\infty$ for $x \in X \setminus A$. Put $\varphi = \mathfrak{I}_A - f$. Then φ is a lower semi-continuous bounded below function and so by the Ekeland variational principle there is $x_0 \in X$ such that $\varphi(x) \ge \varphi(x_0) - \varepsilon\|x - x_0\|$ for every $x \in X$. Obviously $x_0 \in A$ and for every $x \in A$ we have $-f(x) \ge -f(x_0) - \varepsilon\|x - x_0\|$ from which it follows that

$$f(x - x_0) \le \varepsilon\|x - x_0\| \quad \text{for every } x \in A. \tag{1}$$

Let $\delta > 0$ and $Z \in \mathcal{Z}$ be such that A is determined by Z on $U(x_0, \delta)$. For any $z \in Z$, $\|z\| < \delta$ we have $x_0 + z \in A$ and hence $f(z) \le \varepsilon\|z\|$ by (1). This means that $\|f\restriction_Z\| \le \varepsilon$. By the Hahn-Banach theorem we can find $g \in X^*$ such that $g = f$ on Z and $\|g\| \le \varepsilon$. Clearly, $f - g \in \mathcal{Z}^{\perp}$ and $\|f - (f - g)\| \le \varepsilon$.

□

The next corollary removes the assumption of continuity in a theorem from [FZ].

Corollary 18. *Let X be a Banach space, $M \subset X^*$ and X admits an arbitrary LFC-M bump function. Then $\overline{\operatorname{span}} M = X^*$.*

Proof. Let b be the LFC-M bump function. Put $A = \{x \in X;\ b(x) \ne 0\}$ and $\mathcal{Z} = \left\{ \bigcap_{i=1}^{n} \ker f_i;\ f_1, \dots, f_n \in M, n \in \mathbb{N} \right\}$. Then A is a non-empty bounded LFD-$\mathcal{Z}$ set (Fact 6), $\mathcal{Z}^{\perp} = \operatorname{span} M$ and so $\overline{\operatorname{span}} M = X^*$ by Theorem 17.

□

Corollary 19. *Any infinite-dimensional locally flat Banach space X is a c_0-saturated Asplund space.*

Proof. X is c_0-saturated by Theorem 16. Since local flatness passes to subspaces, it is enough to show that X^* is separable provided that X is separable.

By the Lindelöf property of X there exists a countable collection $\mathcal{Z} = \{Z_i\} \subset \mathcal{FC}(X)$ such that A is LFD-$\mathcal{Z}$. If $Z \in \mathcal{FC}(X)$, then $Z^\perp \subset X^*$ is a subspace with $\dim Z^\perp \leq \operatorname{codim} Z$. As Z_i is finite-codimensional, we can find $\{f_{i,j}\}_{j=1}^{n_i} \subset Z_i^\perp$, such that $Z_i^\perp = \operatorname{span}\{f_{i,j}\}_{j=1}^{n_i}$, where $n_i \leq \operatorname{codim} Z_i$. We have $\mathcal{Z}^\perp = \bigcup_i Z_i^\perp \subset \operatorname{span} \bigcup_i Z_i^\perp = \operatorname{span} \bigcup_i \{f_{i,1}, \dots, f_{i,n_i}\}$ and by Theorem 17, $X^* = \overline{\operatorname{span}} \bigcup_i \{f_{i,1}, \dots, f_{i,n_i}\}$, hence it is separable. □

3. Spaces with Schauder Bases

The word "coordinate" in the term LFC originates of course from spaces with bases, where LFC was first defined using the coordinate functionals. In order to apply the LFC techniques to spaces without a Schauder basis, the notion had to be obviously generalised using arbitrary functionals from the dual. However, as we will show in this section, the generalisation does not substantially increase the supply of LFC functions on Banach spaces with a Schauder basis, and we can always in addition assume that the given LFC function in fact depends on the coordinate functionals. This fact is not only interesting in itself; it is the main tool for smoothing up LFC bumps on separable spaces with basis.

We begin with a simple related result for Markushevich bases:

Theorem 20. *Let E be a set, X be a separable Banach space and $g\colon X \to E$ be a LFC mapping. Then there is a Markushevich basis $\{x_i, x_i^*\} \subset X \times X^*$ such that g is LFC-$\{x_i^*\}$.*

Proof. By the Lindelöf property of X we can choose a countable $\{f_i\} \subset X^*$ such that g is LFC-$\{f_i\}$. Find a countable $\{g_i\} \subset X^*$ such that it separates points of X and $\{f_i\} \subset \{g_j\}$. Then we can use the Markushevich theorem (see e.g. [F–Z]) to construct a Markushevich basis $\{x_i, x_i^*\}$ such that $\operatorname{span}\{x_i^*\} = \operatorname{span}\{g_i\} \supset \operatorname{span}\{f_i\}$.

Now let $x \in X$ and $U \subset X$ be a neighbourhood of x such that g depends only on $M = \{f_1, \dots, f_n\}$ on U. Let $M \subset \operatorname{span}\{x_1^*, \dots, x_m^*\}$. Then for any $y, z \in U$ such that $x_j^*(y) = x_j^*(z)$ for all $j = 1, \dots, m$ we have also $f_i(y) = f_i(z)$ for any $i = 1, \dots, n$ and hence $g(y) = g(z)$. Thus g depends only on $\{x_1^*, \dots, x_m^*\}$ on U. □

We would like to establish a similar result for Schauder bases. In this context, shrinking Schauder bases emerge quite naturally, taking into account Corollary 18 (see also Theorem 28). We will use the following simple fact:

Fact 21. *Let X and Y be Banach spaces with equivalent Schauder bases $\{x_i\}$ and $\{y_i\}$ respectively. Then $\{x_i\}$ is shrinking if and only if $\{y_i\}$ is shrinking.*

Proof. Let $\{x_i\}$ be a shrinking basis and $T\colon Y \to X$ be an isomorphism of Y onto X such that $Ty_i = x_i$. Then $T^*\colon X^* \to Y^*$ is an isomorphism of X^* onto Y^* such that $T^*x_i^* = y_i^*$ and thus

$$Y^* = T^*(X^*) = T^*(\overline{\operatorname{span}}\{x_i^*\}) \subset \overline{T^*(\operatorname{span}\{x_i^*\})} = \overline{\operatorname{span}}\, T^*(\{x_i^*\}) = \overline{\operatorname{span}}\{y_i^*\}.$$

□

The next result is the main tool used in the sequel for the study of functions locally dependent on finitely many coordinates on spaces with shrinking Schauder bases.

Lemma 22. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$. Let $f \in X^*$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X and $N \in \mathbb{N}$, $N > n$, such that $x_i = e_i$ for $1 \leq i < N$, $\{x_i\}$ is $(1+\varepsilon)$-equivalent to $\{e_i\}$, $\operatorname{span}\{x_i\}_{i=k}^m = \operatorname{span}\{e_i\}_{i=k}^m$ for all $1 \leq k \leq n$ and $m \geq k$, $x_i^* = e_i^*$ if $i < n$ or $i \geq N$, and $\overline{\operatorname{span}}\{x_i;\ i \geq N\} \subset \ker f$.*

Proof. Without loss of generality we may assume that there is a $z \in \operatorname{span}\{e_i;\ i \geq n\}$ for which $f(z) = 1$. Let us denote $f_k = f - P_{k-1}^* f$. As $\{e_i\}$ is shrinking, $\|f_k\| \to 0$ and hence we can find $N \in \mathbb{N}$ such that $N > \max\operatorname{supp} z \geq n$ and $\|f_N\| \leq \frac{\varepsilon}{(2+\varepsilon)\|z\|}$. Put $x_i = e_i$ for $1 \leq i < N$ and $x_i = e_i - f(e_i)z$ for $i \geq N$. For any m_1,$m_2 \in \mathbb{N}$ and any sequence $\{a_i\}$ of scalars we have

$$\begin{aligned}\left\|\sum_{i=m_1}^{m_2} a_i x_i\right\| &= \left\|\sum_{i=m_1}^{m_2} a_i e_i - z\sum_{i=\max\{m_1,N\}}^{m_2} a_i f(e_i)\right\| \leq \left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| + \left\|z f_N\left(\sum_{i=m_1}^{m_2} a_i e_i\right)\right\| \\ &\leq \big(1 + \|z\|\,\|f_N\|\big)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| \leq \left(1 + \frac{\varepsilon}{2+\varepsilon}\right)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\|\end{aligned}$$

and

$$\left\|\sum_{i=m_1}^{m_2} a_i x_i\right\| \geq \left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| - \left\|z f_N\left(\sum_{i=m_1}^{m_2} a_i e_i\right)\right\| \geq \left(1 - \frac{\varepsilon}{2+\varepsilon}\right)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\|.$$

This implies that $\{x_i\}$ is a basic sequence $(1+\varepsilon)$-equivalent to $\{e_i\}$. Since $z \in \operatorname{span}\{x_i;\ n \le i < N\}$, we have $\operatorname{span}\{x_i\}_{i=k}^m = \operatorname{span}\{e_i\}_{i=k}^m$ for all $1 \le k \le n$ and $m \ge k$, and therefore $\operatorname{span}\{x_i\} = \operatorname{span}\{e_i\}$, which implies that $\{x_i\}$ is a basis of X. Moreover,

$$\begin{aligned} x_i^*(x) &= \sum e_j^*(x)x_i^*(e_j) = \sum_{j<N} e_j^*(x)x_i^*(x_j) + \sum_{j\ge N} e_j^*(x)x_i^*\big(x_j + f(e_j)z\big) \\ &= \sum e_j^*(x)x_i^*(x_j) + x_i^*(z)\sum_{j\ge N} e_j^*(x)f(e_j) = e_i^*(x) \quad \text{if } i<n \text{ or } i \ge N. \end{aligned}$$

Finally, $f(x_i) = 0$ for $i \ge N$.

□

It is perhaps worth noticing that the method used in the previous lemma (and the next theorem) does not rely on the classical argument of perturbation by the norm-summable sequence. In fact our new basis is "far" away from the original one.

Theorem 23. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$, let $\{f_i\} \subset X^*$ be a countable subset and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X such that it is $(1+\varepsilon)$-equivalent to $\{e_i\}$, $\operatorname{span}\{x_i\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$ and $\operatorname{span}\{f_i\} \subset \operatorname{span}\{x_i^*\}$.*

Proof. Choose a sequence of $\varepsilon_i > 0$ such that $\prod_i(1+\varepsilon_i) \le (1+\varepsilon)$ and put $N_0 = 1$. We apply Lemma 22 to $\{e_i\}$, f_1, ε_1 and $n = 1$. We obtain a basis $\{x_i^1\}$ which is $(1+\varepsilon_1)$-equivalent to $\{e_i\}$ and $N_1 \in \mathbb{N}$ such that $\overline{\operatorname{span}}\{x_i^1;\ i \ge N_1\} \subset \ker f_1$. Moreover, $x_i^1 = e_i$ for $i < N_1$ and $\operatorname{span}\{x_i^1\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$.

We proceed by induction. Suppose the basis $\{x_i^k\}$ and $N_k \in \mathbb{N}$ have already been defined in such a way that the basis $\{x_i^k\}$ is $\prod_{j\le k}(1+\varepsilon_j)$-equivalent to $\{e_i\}$, $x_i^k = x_i^{k-1}$ for $i < N_k$, $\operatorname{span}\{x_i^k\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$, and finally $\overline{\operatorname{span}}\{x_i^k;\ i \ge N_j\} \subset \ker f_j$ for $1 \le j \le k$. We apply Lemma 22 to $\{x_i^k\}$, f_{k+1}, ε_{k+1} and $n = N_k$ in order to obtain a basis $\{x_i^{k+1}\}$ which is $\prod_{j\le k+1}(1+\varepsilon_j)$-equivalent to $\{e_i\}$ and a number $N_{k+1} \in \mathbb{N}$, $N_{k+1} > N_k$, such that $\overline{\operatorname{span}}\{x_i^{k+1};\ i \ge N_{k+1}\} \subset \ker f_{k+1}$. Moreover, $x_i^{k+1} = x_i^k$ for $i < N_{k+1}$ and $\operatorname{span}\{x_i^{k+1}\}_{i=1}^m = \operatorname{span}\{x_i^k\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$. Since also $\operatorname{span}\{x_i^{k+1}\}_{i=N_j}^m = \operatorname{span}\{x_i^k\}_{i=N_j}^m$ for all $1 \le j \le k$ and $m \ge N_j$, we have $\overline{\operatorname{span}}\{x_i^{k+1};\ i \ge N_j\} \subset \ker f_j$ for $1 \le j \le k+1$.

Clearly, there is a sequence $\{x_i\}$ such that $\lim_{j\to\infty} x_i^j = x_i$ for all $i \in \mathbb{N}$. (This is because the sequence N_k is increasing and thus x_i^j is eventually constant (in j).) It is straightforward to check that $\operatorname{span}\{x_i\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$, $\{x_i\}$ is a basis of X which is $(1+\varepsilon)$-equivalent to $\{e_i\}$ and $\overline{\operatorname{span}}\{x_i;\ i \ge N_j\} \subset \ker f_j$ (which means that $f_j \in \operatorname{span}\{x_i^*;\ i < N_j\}$) for any $j \in \mathbb{N}$.

□

If a Banach space X has a shrinking Schauder basis, using the Lindelöf property of X (as in the proof of Theorem 20) and Theorem 23 we obtain the following corollary, which allows us to work only with LFC-$\{e_i^*\}$ functions.

Corollary 24. *Let E be a set, X be a Banach space with a shrinking Schauder basis $\{e_i\}$, $g\colon X \to E$ be a LFC mapping and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X, $(1+\varepsilon)$-equivalent to $\{e_i\}$, such that g is LFC-$\{x_i^*\}$.*

Using Fact 6 we can reformulate this corollary as follows:

Corollary 25. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$, $A \subset X$ be LFD-$\mathcal{FC}_c(X)$ and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X, $(1+\varepsilon)$-equivalent to $\{e_i\}$, such that A is LFD-$\mathcal{Z}$ for $\mathcal{Z} = \{\overline{\operatorname{span}}\{x_i\}_{i=n}^\infty;\ n \in \mathbb{N}\}$.*

The following lemma seems to be the crucial reason why we need to work with Schauder bases.

Lemma 26. *Let X be a Banach space with a Schauder basis $\{e_i\}$ and E be an arbitrary set. Then $f : X \to E$ is LFC-$\{e_i^*\}$ if and only if for each $x \in X$ there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $f(y) = f(P_n y)$ whenever $\|x - y\| < \delta$ and $n \ge n_0$.*

Proof. The "if" part is trivial: $P_{n_0}y = P_{n_0}z$ whenever $e_i^*(y) = e_i^*(z)$ for $1 \le i \le n_0$. Thus $f(y) = f(P_{n_0}y) = f(P_{n_0}z) = f(z)$ if moreover $y, z \in U(x,\delta)$, which means that f depends only on $\{e_1^*, \dots, e_{n_0}^*\}$ on $U(x,\delta)$.

The "only if" part is also simple. Let K be a basis constant of $\{e_i\}$ and $x \in X$. There is $m \in \mathbb{N}$ and $\delta > 0$ such that $f(y) = f(z)$ if $y, z \in U(x, \delta(1+K))$ and $e_i^*(y) = e_i^*(z)$ for $1 \le i \le m$. Choose $n_0 \ge m$ such that $\|x - P_n x\| < \delta$ for all $n \ge n_0$. Then for any $n \ge n_0$ and $y \in X$ such that $\|x - y\| < \delta$ we have $\|P_n y - x\| \le \|P_n y - P_n x\| + \|P_n x - x\| < \delta(1+K)$ and therefore $f(y) = f(P_n y)$.

□

Let X be a Banach lattice. We say that a function $f : X \to \mathbb{R}$ is a lattice function if it satisfies either $f(x) \le f(y)$ whenever $|x| \le |y|$, or $f(x) \ge f(y)$ whenever $|x| \le |y|$. Recall that a Banach space X with an unconditional basis $\{e_i\}$ has a natural lattice structure defined by $\sum a_i e_i \ge 0$ if and only if $a_i \ge 0$ for all $i \in \mathbb{N}$. The same holds for ℓ_∞.

The following technical lemma will be useful later for smoothing up lattice functions.

Lemma 27. *Let $f : \mathbb{R} \to \mathbb{R}$ be an even function that is non-decreasing on $[0,\infty)$ and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an even function with bounded support that is non-increasing on $[0,\infty)$. Then $(f * \varphi)(x) = \int_{\mathbb{R}} f(x-t)\varphi(t)\,dt$ is an even function that is non-decreasing on $[0,\infty)$.*

Proof. Note that $f * \varphi$ is well defined as f and φ are bounded on bounded sets.

Obviously, $(f * \varphi)(-x) = \int_{\mathbb{R}} f(-x-t)\varphi(t)\,dt = \int_{\mathbb{R}} f(x+t)\varphi(t)\,dt = \int_{\mathbb{R}} f(x-t)\varphi(t)\,dt = (f * \varphi)(x)$, using first the fact that f is even, then the fact that φ is even.

Now pick any $0 \le x < y < \infty$. The function $\psi(t) = \varphi(\frac{y-x}{2} - t) - \varphi(\frac{x-y}{2} - t)$ is an odd function (this is obvious), such that $\psi(t) \ge 0$ for $t \ge 0$. Indeed, either we have $0 \ge \frac{y-x}{2} - t \ge \frac{x-y}{2} - t$, or $0 < \frac{y-x}{2} - t \le t - \frac{x-y}{2}$ and in both cases we use the properties of φ. Similarly we get that the function $t \mapsto f\left(\frac{x+y}{2} + t\right) - f\left(\frac{x+y}{2} - t\right)$ is non-negative for $t \ge 0$. Therefore,

$$
\begin{aligned}
(f * \varphi)(y) - (f * \varphi)(x) &= \int_{\mathbb{R}} f(t)\bigl(\varphi(y-t) - \varphi(x-t)\bigr)\,dt = \int_{\mathbb{R}} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\
&= \int_{(-\infty,0)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt + \int_{(0,\infty)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\
&= -\int_{(0,\infty)} f\left(\tfrac{x+y}{2} - t\right)\psi(t)\,dt + \int_{(0,\infty)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\
&= \int_{(0,\infty)} \left(f\left(\tfrac{x+y}{2} + t\right) - f\left(\tfrac{x+y}{2} - t\right)\right)\psi(t)\,dt \ge 0.
\end{aligned}
$$

□

Now we can prove one of the main results of this paper.

Theorem 28. *Let X be a Banach space with a Schauder basis $\{e_i\}$. The following statements are equivalent:*

(i) $\{e_i\}$ is shrinking and X admits a continuous LFC bump.
(ii) X admits a continuous LFC-$\{e_i^*\}$ bump.
(iii) X admits a C^∞-smooth LFC-$\{e_i^*\}$ bump.

For the proof of Theorem 28 we will need the following lemma, the basic idea of which is implicitly contained in [Haj1]. Let $\Delta = \{\delta_n\}_{n=1}^\infty$ be a sequence of positive real numbers. We denote by V^Δ an open subset of ℓ_∞ such that $x \in V^\Delta$ if and only if there is $n_x \in \mathbb{N}$ satisfying $|x(n_x)| - \delta_{n_x} > \sup_{n>n_x} |x(n)| + \delta_{n_x}$. For any $x \in V^\Delta$, the set

$$
V_{n_x}^\Delta = \left\{ y \in \ell_\infty \colon\ |y(n_x)| - \delta_{n_x} > \sup_{n>n_x} |y(n)| + \delta_{n_x} \right\} \subset V^\Delta
$$

is an open neighbourhood of x in ℓ_∞.

Lemma 29. *Let $\varepsilon > 0$ and a sequence $\Delta = \{\delta_n\}_{n=1}^\infty$, $\delta_n > 0$ be given. There is a convex lattice 1-Lipschitz function $F \colon \ell_\infty \to \mathbb{R}$ such that $\|x\|_\infty \le F(x) \le \|x\|_\infty + \varepsilon$ for any $x \in \ell_\infty$ and F is LFC-$\{e_i^*\}$ and C^∞ on V^Δ. Moreover, for any $x \in V^\Delta$, F depends only on $\{e_i^*\}_{i=1}^{n_x}$ on $V_{n_x}^\Delta$, where e_i^* are the coordinate functionals on ℓ_∞.*

Proof. Let $\varepsilon_1 = \min\{\delta_1, \varepsilon\}$ and $\varepsilon_n = \min\{\delta_n, \varepsilon_{n-1}\}$ for $n > 1$. Choose a sequence $\{\varphi_n\}_{n=1}^\infty$ of C^∞-smooth even functions $\varphi_n \colon \mathbb{R} \to [0,\infty)$ such that $\operatorname{supp} \varphi_n \subset [-\varepsilon_n, \varepsilon_n]$, φ_n is non-increasing on $[0,\infty)$ and $\int_{\mathbb{R}} \varphi_n(t)\,dt = 1$. Define a sequence $\{F_n\}_{n=0}^\infty$ of functions $F_n \colon \ell_\infty \to \mathbb{R}$ by the inductive formula

$$
\begin{aligned}
F_0(x) &= \|x\|_\infty, \\
F_n(x) &= \int_{\mathbb{R}} F_{n-1}(x + te_n)\varphi_n(t)\,dt.
\end{aligned}
$$

It is easily checked that each F_n is convex, 1-Lipschitz and $F_n(x) - \|x\|_\infty \le \varepsilon$ for any $x \in \ell_\infty$. To see that F_n is lattice, pick $x, y \in \ell_\infty$, $x = (x_i)$, $y = (y_i)$, satisfying $|y| \le |x|$. Define $g \colon \mathbb{R} \to \mathbb{R}$ by $g(u) = F_{n-1}(y + (u - y_n)e_n)$. Then

$$
\begin{aligned}
F_n(x) - F_n(y) &= \int_{\mathbb{R}} \bigl(F_{n-1}(x + te_n) - F_{n-1}(y + te_n)\bigr)\varphi_n(t)\,dt \\
&= \int_{\mathbb{R}} \Bigl(F_{n-1}(x + te_n) - F_{n-1}\bigl(y + (x_n - y_n + t)e_n\bigr)\Bigr)\varphi_n(t)\,dt \\
&\qquad + \int_{\mathbb{R}} \Bigl(F_{n-1}\bigl(y + (x_n - y_n + t)e_n\bigr) - F_{n-1}(y + te_n)\Bigr)\varphi_n(t)\,dt \\
&= \int_{\mathbb{R}} \Bigl(F_{n-1}(x + te_n) - F_{n-1}\bigl(y + (x_n - y_n + t)e_n\bigr)\Bigr)\varphi_n(t)\,dt + g * \varphi_n(x_n) - g * \varphi_n(y_n) \ge 0,
\end{aligned}
$$

because $F_{n-1}(x + te_n) \ge F_{n-1}(y + (x_n - y_n + t)e_n)$ which follows from the induction hypothesis (notice that we have $x + te_n = (x_1, \dots, x_{n-1}, x_n + t, x_{n+1}, \dots)$ and $y + (x_n - y_n + t)e_n = (y_1, \dots, y_{n-1}, x_n + t, y_{n+1}, \dots)$, thereby $|x + te_n| \ge$

$|y+(x_n-y_n+t)e_n|$ in the lattice sense), and because g is an even function non-decreasing on $[0,\infty)$ also by the induction hypothesis and so we may use Lemma 27.

Further, by Jensen's inequality,

$$F_n(x)=\int_{\mathbb{R}} F_{n-1}(x+te_n)\varphi_n(t)\,dt \geq F_{n-1}\left(x+e_n\int_{\mathbb{R}} t\varphi_n(t)\,dt\right)=F_{n-1}(x),$$

which means that the sequence $\{F_n\}$ is non-decreasing. Consequently the function $F=\lim_n F_n=\sup_n F_n$ is convex, lattice, 1-Lipschitz and $\|x\|_\infty\leq F(x)\leq\|x\|_\infty+\varepsilon$ for any $x\in\ell_\infty$.

For any $y\in\ell_\infty$ and $k\in\mathbb{N}$ we have

$$F_k(y)=\int_{-\varepsilon_k}^{\varepsilon_k}\cdots\int_{-\varepsilon_1}^{\varepsilon_1}\|y+t_1e_1+\cdots+t_ke_k\|_\infty\,\varphi_1(t_1)\cdots\varphi_k(t_k)\,dt_1\cdots dt_k.$$

Fix an arbitrary $x\in V^\Delta$ and pick any $y\in V^\Delta_{n_x}$ and $k>n_x$. Then

$$\|y+t_1e_1+\cdots+t_ke_k\|_\infty=\|y+t_1e_1+\cdots+t_{n_x}e_{n_x}\|_\infty=\|P_{n_x}y+t_1e_1+\cdots+t_{n_x}e_{n_x}\|_\infty,$$

as long as $|t_i|\leq\delta_{n_x}$ for $n_x\leq i\leq k$. Since $\varepsilon_n\leq\delta_{n_x}$ for $n\geq n_x$ and $\int_{\mathbb{R}}\varphi_n=1$, it follows that $F_k(y)=F_{n_x}(y)=F_{n_x}(P_{n_x}y)$. This means that $F(y)=F_{n_x}(P_{n_x}y)$ and therefore F is C^∞-smooth and depends only on $\{e_i^*\}_{i=1}^{n_x}$ on $V^\Delta_{n_x}$.

□

Proof of Theorem 28. (iii)⇒(i) follows from Corollary 18.

(i)⇒(ii) follows from Corollary 24: If g is a continuous LFC bump on X, let $\{x_i\}$ be a basis obtained from Corollary 24 and T be an isomorphism $e_i\mapsto x_i$. Then the function $g\circ T$ is a continuous LFC-$\{e_i^*\}$ bump.

It remains to prove (ii)⇒(iii). Since X admits a continuous LFC-$\{e_i^*\}$ bump, using an affine transformation and a composition with a suitable function we can produce a continuous LFC-$\{e_i^*\}$ function $b\colon X\to[1,2]$ such that $b(0)=1$ and $b(x)=2$ whenever $\|x\|\geq 1$. Choose a sequence of real numbers $\{\eta_n\}$ decreasing to 1 such that $\eta_1<1+\frac14$ and a decreasing sequence $\Delta=\{\delta_n\}$ such that $0<\delta_n<\frac14(\eta_n-\eta_{n+1})$ and $\delta_1<\frac18$.

For a fixed $n\in\mathbb{N}$, let $T_n\colon\mathbb{R}^n\to P_nX$ be a canonical isomorphism, i.e. $T_n(t_1,\dots,t_n)=t_1e_1+\cdots+t_ne_n$. Because $b\circ T_n\in C(\mathbb{R}^n)$ and it is constant outside a sufficiently large ball in $\mathbb{R}^n$, using standard finite-dimensional smooth approximations we can find $\widetilde{b}_n\in C^\infty(\mathbb{R}^n)$ such that $\sup_{\mathbb{R}^n}|\widetilde{b}_n(y)-\eta_nb(T_ny)|<\delta_n$. We define $b_n(x)=\widetilde{b}_n(T_n^{-1}P_nx)$ and thus $b_n\in C^\infty(X)$ and $\sup_X|b_n(x)-\eta_nb(P_nx)|<\delta_n$.

Further, let us define $\Phi\colon X\to\ell_\infty$ by $\Phi(x)(n)=b_n(x)$. Pick any $x\in X$. By Lemma 26 there is $\delta>0$ and $n_x\in\mathbb{N}$ such that $b(y)=b(P_ny)$ whenever $\|x-y\|<\delta$ and $n\geq n_x$. Thus for $n>m\geq n_x$ and $\|x-y\|<\delta$ we have

$$\begin{aligned}|\Phi(y)(m)|-\delta_m&=b_m(y)-\delta_m>\eta_mb(P_my)-2\delta_m=\eta_mb(y)-2\delta_m>\eta_{m+1}b(y)+2\delta_m\\&>\eta_nb(y)+\delta_n+\delta_m=\eta_nb(P_ny)+\delta_n+\delta_m>b_n(y)+\delta_m=|\Phi(y)(n)|+\delta_m.\end{aligned}$$

(The second inequality follows from the definition of δ_m.) It means that $|\Phi(y)(n_x)|-\delta_{n_x}>|\Phi(y)(n_x+1)|+\delta_{n_x}=\sup_{n>n_x}|\Phi(y)(n)|+\delta_{n_x}$. As $x\in X$ is arbitrary, these inequalities show that $\Phi(X)\subset V^\Delta$ and moreover

$$\Phi(y)\in V^\Delta_{n_x}\quad\text{whenever } \|x-y\|<\delta. \tag{2}$$

We now apply Lemma 29 to the sequence Δ and $\varepsilon<\frac18$ in order to obtain the corresponding function F and we set $f=F\circ\Phi$. The properties of F together with (2) and the fact that b_n depends only on $\{e_i^*\}_{i=1}^n$ imply that f is LFC-$\{e_i^*\}$. Moreover, as $F\circ\Phi=F\circ P_{n_x}\circ\Phi$ on a neighbourhood of x and $P_{n_x}\circ\Phi\in C^\infty(X,\ell_\infty)$, we can conclude that f is C^∞-smooth.

Further,

$$f(0)=F\big(\Phi(0)\big)\leq\|\Phi(0)\|_\infty+\varepsilon=\sup_n b_n(0)+\varepsilon\leq\sup_n\big(\eta_nb(0)+\delta_n\big)+\varepsilon=\eta_1+\delta_1+\varepsilon<1+\frac12.$$

On the other hand, if $\|x\|\geq1$ we get

$$f(x)\geq\|\Phi(x)\|_\infty=\sup_n b_n(x)\geq b_{n_x}(x)>\eta_{n_x}b(P_{n_x}x)-\delta_{n_x}=\eta_{n_x}b(x)-\delta_{n_x}>2-\delta_1>2-\frac18.$$

Therefore f is a separating function on X and we obtain the desired bump by composing f with a suitable smooth real function.

□

Theorem 30. *Let X be a Banach space with an unconditional Schauder basis $\{e_i\}$, which admits a continuous LFC bump. Then X admits a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump.*

Proof. Since X is c_0-saturated (Theorem 16 or [PWZ]), it does not contain ℓ_1 and so by James's theorem $\{e_i\}$ is shrinking. By Theorem 28 there is a continuous LFC-$\{e_i^*\}$ bump b on X and without loss of generality we may assume $b\colon X\to[0,1]$ and $b(0)>0$. We may further assume that the norm $\|\cdot\|$ on X is lattice.

First we show that there is a continuous lattice LFC-$\{e_i^*\}$ bump on X. Put $g(x) = \inf_{|y|\leq|x|} b(y)$. If $b(x) = 0$ then also $g(x) = 0$ and $g(0) = b(0) > 0$, hence g is a bump function. Further, for any $x, y \in X$ such that $|y| \leq |x|$ we have $g(y) = \inf_{|z|\leq|y|} b(z) \geq \inf_{|z|\leq|x|} b(z) = g(x)$, thus g is lattice.

For any $y \in X$ we denote $y(i) = e_i^*(y)$. Define a mapping $\psi\colon X \to 2^X$ by $\psi(y) = \{z \in X;\ |z| \leq |y|\}$. Clearly, $\psi(y)$ is a convex set for any $y \in X$. Furthermore, as $\{e_i\}$ is unconditional, $\psi(y)$ is a compact set for any $y \in X$ (consider the mapping from a compact space $\prod_i [-|y(i)|, |y(i)|]$ into X defined by $(t_1, t_2, \dots) \mapsto \sum t_i e_i$).

Now fix an arbitrary $x \in X$. Let us define a projection $y \mapsto \tilde{y}$ from X onto $\psi(x)$: For any $y \in X$ we put $\tilde{y}(i) = y(i)$ if $|y(i)| \leq |x(i)|$, $\tilde{y}(i) = \operatorname{sgn} y(i)\,|x(i)|$ otherwise. Notice that $|\tilde{y}| \leq |x|$ and so indeed $\tilde{y} \in \psi(x)$. Let $z \in X$. Then $\|y - \tilde{y}\| \leq \|x - z\|$ for any $y \in \psi(z)$. Indeed, $|\tilde{y}(i) - y(i)| = \big|\operatorname{sgn} y(i)\,|x(i)| - y(i)\big| = \big||x(i)| - |y(i)|\big| = |y(i)| - |x(i)| \leq |z(i)| - |x(i)| \leq |z(i) - x(i)|$ whenever $|y(i)| > |x(i)|$. Thus $|\tilde{y} - y| \leq |x - z|$ and we use the fact that $\|\cdot\|$ is lattice.

Let U be a neighbourhood of $\psi(x)$ and $\delta = \operatorname{dist}(\psi(x), X \setminus U)$. Suppose $z \in X$, $\|x - z\| < \delta$. Then $\|y - \tilde{y}\| \leq \|x - z\| < \delta$ for any $y \in \psi(z)$ and hence $\psi(z) \subset U$. This implies that ψ is a cusco mapping.

Given any $\varepsilon > 0$ we can find a neighbourhood U of $\psi(x)$ and $0 < \delta < \operatorname{dist}(\psi(x), X \setminus U)$ such that $|b(y) - b(z)| < \varepsilon$ whenever $y, z \in U$, $\|y - z\| < \delta$. Suppose $z \in X$, $\|x - z\| < \delta$. Then, by the previous paragraph, $|b(\tilde{y}) - b(y)| < \varepsilon$. Therefore, $g(z) = \inf_{y\in\psi(z)} b(y) \geq \inf_{y\in\psi(z)} b(\tilde{y}) - \varepsilon \geq \inf_{y\in\psi(x)} b(y) - \varepsilon = g(x) - \varepsilon$. Similarly, considering a projection onto $\psi(z)$, we obtain $g(x) \geq g(z) - \varepsilon$. This shows that g is continuous.

Suppose that for some $F \subset \mathbb{N}$ we have $x(i) = y(i)$ for all $i \in F$ and let $w \in \psi(x)$. Define $z \in X$ such that $z(i) = w(i)$ for $i \in F$ and $z(i) = y(i)$ otherwise. Then $z \in \psi(y)$ and the assumption of Lemma 10 is satisfied. Hence g is LFC-$\{e_i^*\}$.

We note that the process described above does not preserve smoothness as can be easily seen on a one-dimensional example.

Finally, we smoothen up the bump g by repeating the proof of Theorem 28. Notice only that the finite-dimensional smooth approximations can be made lattice similarly as in the proof of Lemma 29, consequently $\Phi(\cdot)(n)$ is lattice for each $n \in \mathbb{N}$ and since F from Lemma 29 is lattice too, we can conclude that the resulting function $f = F \circ \Phi$ is lattice.

□

4. Spaces with symmetric Schauder bases

Let X be a Banach space with a symmetric Schauder basis. In such spaces it is possible to define a notion of the *non-increasing reordering*, which will be one of the main tools in the sequel. For any $x \in X$, $x = (x_i)$, let us denote by $\widehat{x}$ a vector in X with its coordinates formed by the non-increasing reordering of the sequence $(|x_i|)$. Notice that we can view X as a linear subspace of c_0 through the natural "coordinate" embedding. In the following lemma we gather some simple properties of this reordering which will be used later.

Lemma 31. *Let X be a Banach space with a symmetric Schauder basis, $x, y \in X$ be arbitrary.*

(a) Let $\|\cdot\|$ be a symmetric lattice norm on X. Then $\big|\|P_k\widehat{x}\| - \|P_k\widehat{y}\|\big| \leq \|x - y\|$ for any $k \in \mathbb{N}$.
(b) $\widehat{R_n\widehat{x}} \leq \widehat{R_n x}$ in the lattice sense for any $n \in \mathbb{N}$.
(c) $\|\widehat{x} - \widehat{y}\|_\infty \leq \|x - y\|_\infty$.
(d) Let $\|\cdot\|$ be a lattice norm on X such that the basis is normalised. Then the mapping $x \mapsto P_n\widehat{x}$ is n-Lipschitz for any $n \in \mathbb{N}$.

Proof. (a): Consider a set $A \subset \mathbb{N}$, $|A| = k$, such that $\widehat{P_A x} = P_k\widehat{x}$. Since $\|\cdot\|$ is symmetric and lattice, $\|P_k\widehat{x}\| = \|P_A x\|$ and $\|P_k\widehat{y}\| \geq \|P_A y\|$. Therefore $\|P_k\widehat{x}\| - \|P_k\widehat{y}\| \leq \|P_A x\| - \|P_A y\| \leq \|P_A(x - y)\| \leq \|x - y\|$.

(b): Let $A \subset \mathbb{N}$, $|A| \leq n$ be such that $\widehat{R_n x} = \widehat{w}$, where $w = \widehat{x} - P_A\widehat{x}$. We put $z = R_n\widehat{x}$. Then $\widehat{z}_i = \widehat{x}_{i+n}$ for $i \in \mathbb{N}$. Let $\pi\colon \mathbb{N} \to \mathbb{N}$ be a one to one mapping such that $\widehat{w}_i = w_{\pi(i)}$. Then $\widehat{w}_i = \widehat{x}_{\pi(i)}$ for $i \in \mathbb{N}$. As $i \leq \pi(i) \leq i + n$, it follows that $\widehat{z}_i = \widehat{x}_{i+n} \leq \widehat{x}_{\pi(i)} = \widehat{w}_i$.

(c): Let $\pi\colon \mathbb{N} \to \mathbb{N}$ and $\sigma\colon \mathbb{N} \to \mathbb{N}$ be one to one mappings such that $\widehat{x}_i = |x_{\pi(i)}|$ and $\widehat{y}_i = |y_{\sigma(i)}|$. Pick any $n \in \mathbb{N}$. There is $k \leq n$ such that $|y_{\pi(k)}| \leq |y_{\sigma(n)}|$. (Otherwise there would be at least n coordinates of y for which their absolute value is greater than $|y_{\sigma(n)}|$ which is impossible.) Consequently, $\widehat{x}_n - \widehat{y}_n = |x_{\pi(n)}| - |y_{\sigma(n)}| \leq |x_{\pi(k)}| - |y_{\pi(k)}| \leq |x_{\pi(k)} - y_{\pi(k)}| \leq \|x - y\|_\infty$.

(d): Using the fact that the basis is normalised, then (c) and then the fact that $\|\cdot\|$ is lattice we obtain $\|P_n\widehat{x} - P_n\widehat{y}\| = \|P_n(\widehat{x} - \widehat{y})\| \leq \sum_{i=1}^n |(\widehat{x} - \widehat{y})_i| \leq n\,\|\widehat{x} - \widehat{y}\|_\infty \leq n\,\|x - y\|_\infty \leq n\,\|x - y\|$.

□

This is the key lemma:

Lemma 32. *Let X be a Banach space with a symmetric Schauder basis $\{e_i\}$, $\Phi\colon X \to \mathbb{R}$ be a continuous function such that $\Phi(x) > 0$ if $x \neq 0$ and $\{\gamma_n\} \subset (0, +\infty)$ be a decreasing sequence. For any $N \in \mathbb{N}$ define*

$$\Psi_N(x) = \max_{1\leq n\leq N} \gamma_n \Phi(P_n\widehat{x}).$$

Then each function Ψ_N is LFC-$\{e_i^*\}$ on $X \setminus \{0\}$.

Proof. Without loss of generality we may assume that $\|\cdot\|$ is symmetric and lattice. Let $N \in \mathbb{N}$ and $x \in X \setminus \{0\}$ be given. We claim that there exist a neighbourhood V of x and $N_1 \in \mathbb{N}$ such that $\widehat{x}_{N_1} > \widehat{x}_{N_1+1}$ and $\Psi_N(y) = \Psi_{\min\{N,N_1\}}(y)$ for all $y \in V$. If $|\operatorname{supp} x| \geq N$, then there exists $N_1 \geq N$ such that $\widehat{x}_{N_1} > \widehat{x}_{N_1+1}$ and the claim follows. Otherwise, find $N_1 < N$ such that

$\widehat{x}_{N_1} > \widehat{x}_{N_1+1} = 0$. Then choose $0 < \delta < \widehat{x}_{N_1}/2$ such that

$$|\Phi(z) - \Phi(\widehat{x})| < \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\Phi(\widehat{x})$$

if $\|z - \widehat{x}\| < (N_1 + 1)\delta$. Denote $B = \operatorname{supp} x$ and notice that $|B| = N_1$. If $\|x - y\| < \delta$, $i \in B$ and $j \notin B$, then

$$|y_i| \geq |x_i| - \delta \geq \widehat{x}_{N_1} - \delta > 2\delta - \delta = \delta = |x_j| + \delta \geq |y_j|$$

and hence

$$\|R_{N_1}\widehat{y}\| = \|P_{\mathbb{N}\setminus B}\, y\| = \|P_{\mathbb{N}\setminus B}(y - x)\| \leq \|y - x\| < \delta.$$

Thus, for any $n \geq N_1$,

$$\|P_n\widehat{y} - \widehat{x}\| = \|P_n\widehat{y} - P_{N_1}\widehat{x}\| \leq \|R_{N_1}\widehat{y}\| + \|P_{N_1}\widehat{y} - P_{N_1}\widehat{x}\| < \delta + N_1\|\widehat{y} - \widehat{x}\|_\infty \leq \delta + N_1\|\widehat{y} - \widehat{x}\| < (N_1 + 1)\delta.$$

(For the last but one inequality use Lemma 31(c).) It follows from the choice of δ that for $n > N_1$ we have

$$\gamma_n\Phi(P_n\widehat{y}) < \gamma_n\left(1 + \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\right)\Phi(\widehat{x}) \leq \gamma_{N_1+1}\left(1 + \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_{N_1+1}}\right)\Phi(\widehat{x}) = \frac{\gamma_{N_1} + \gamma_{N_1+1}}{2}\Phi(\widehat{x}).$$

On the other hand,

$$\gamma_{N_1}\Phi(P_{N_1}\widehat{y}) > \gamma_{N_1}\left(1 - \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\right)\Phi(\widehat{x}) \geq \gamma_{N_1}\left(1 - \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_{N_1}}\right)\Phi(\widehat{x}) = \frac{\gamma_{N_1} + \gamma_{N_1+1}}{2}\Phi(\widehat{x}).$$

This means that $\Psi_N(y) = \max_{1\leq n\leq N_1}\gamma_n\Phi(P_n\widehat{y})$ for $\|x - y\| < \delta$, which proves the claim.

Using N_1 and V from the claim, let $\varepsilon = (\widehat{x}_{N_1} - \widehat{x}_{N_1+1})/2$. Choose $A \subset \mathbb{N}$, $|A| = N_1$, such that $P_{N_1}\widehat{x} = \widehat{P_A x}$. If $\|x - y\| < \varepsilon$, then $|y_i| > |y_j|$ whenever $i \in A$ and $j \notin A$. Hence for $1 \leq n \leq N_1$ the mappings $y \mapsto P_n\widehat{y}$ depend only on $\{e_i^*\}_{i\in A}$ on $U(x, \varepsilon)$. By the choice of N_1, it follows that Ψ_N depends only on $\{e_i^*\}_{i\in A}$ on $V \cap U(x, \varepsilon)$.

□

5. Orlicz Sequence Spaces

This section contains the main result of the paper, namely a construction of an Orlicz sequence space h_M with a C^∞-smooth and LFC bump, which does not embed into any $C(K)$ space, K scattered compact. As explained in the introduction, our space is possibly non-polyhedral. If so, it would be the first separable example of a Banach space for which the best smoothness (in the wider sense) of its bumps exceeds the best smoothness of its renormings. Indeed, our space has C^∞-smooth renormings, but, if non-polyhedral, it would have no LFC renormings. Up to now, the only examples (due to Haydon [Hay3], see also [DGZ]) with a similar property are non-separable. Recall that Haydon's space has a C^∞-smooth bump, but no equivalent Gâteaux smooth norm (and in fact using basically the same proof one can conclude that it neither has an equivalent LFC renorming).

For the basic properties of Orlicz sequence spaces we refer e.g. to [LT].

Let M be a non-degenerate Orlicz function and denote by h_M the respective Orlicz sequence space. Let us define a function $\nu\colon h_M \to [0, \infty)$ by $\nu(x) = \sum_{i=1}^\infty M(|x_i|)$. It is easily checked that this function is convex, symmetric and lattice, $\nu(0) = 0$, $\nu(x) > 0$ for $x \neq 0$, and, by the definition of the norm in h_M, $\|x\| = 1$ if and only if $\nu(x) = 1$. It follows from the convexity that $\nu(x) \leq \|x\|$ for $x \in B_{h_M}$, while $\nu(x) \geq \|x\|$ if $\|x\| \geq 1$.

Lemma 33. *The mapping $\mu\colon h_M \to \ell_1$ defined by $\mu(x) = \big(M(|x_i|)\big)$ is continuous. Thus the function $\nu(x) = \|\mu(x)\|_{\ell_1}$ is continuous.*

Proof. Suppose $x \in h_M$ and $0 < \varepsilon < 1$. Choose $N \in \mathbb{N}$ such that $\|R_N x\| < \varepsilon/2$. Then, by the continuity of M, we can choose $0 < \delta < \varepsilon/2$ such that $\|P_N(\mu(x) - \mu(y))\|_{\ell_1} = \sum_{i=1}^N \big|M(|x_i|) - M(|y_i|)\big| < \varepsilon$ if $\|x - y\| < \delta$. Further, if $\|x - y\| < \delta$, then $\|R_N y\| \leq \|R_N x\| + \|R_N(x - y)\| \leq \|R_N x\| + \|x - y\| < \varepsilon$ and hence

$$\begin{aligned}\|\mu(x) - \mu(y)\|_{\ell_1} &\leq \|P_N(\mu(x) - \mu(y))\|_{\ell_1} + \|R_N\mu(x)\|_{\ell_1} + \|R_N\mu(y)\|_{\ell_1}\\ &\leq \varepsilon + \nu(R_N x) + \nu(R_N y) \leq \varepsilon + \|R_N x\| + \|R_N y\| < 3\varepsilon.\end{aligned}$$

□

Let M be a non-degenerate Orlicz function such that there is a $K > 1$ for which $\lim_{t\to 0+} M(Kt)/M(t) = \infty$. Leung in [L1] constructs a sequence $\{\eta_k\}$ of real numbers decreasing to 1 such that the norm on h_M defined by $|||x|||_1 = \sup_k \eta_k\,\|P_k\widehat{x}\|$ has the property that for each $x \in h_M$ there is $j \in \mathbb{N}$ such that $|||x|||_1 = |||P_j x|||_1$ and the supremum is attained at some $n \in \mathbb{N}$. An immediate consequence of this is that the norm $|||x||| = \sup_k \eta_k^2\,\|P_k\widehat{x}\|$ is LFC-$\{e_i^*\}$. To see this, fix $x \in h_M \setminus \{0\}$ and let $n \in \mathbb{N}$ be such that $\eta_n\,\|P_n\widehat{x}\| = \sup_k \eta_k\,\|P_k\widehat{x}\|$. Let $\varepsilon = \eta_n\,\|P_n\widehat{x}\|\,(\eta_n - \eta_{n+1})/(\eta_n^2 + \eta_{n+1}^2)$ and take $y \in h_M$ satisfying $\|x - y\| < \varepsilon$. Then, by Lemma 31(a), $\big|\|P_k\widehat{x}\| - \|P_k\widehat{y}\|\big| < \varepsilon$ for any $k \in \mathbb{N}$. Thus, for $k > n$,

$$\eta_n^2\,\|P_n\widehat{y}\| > \eta_n^2\,\|P_n\widehat{x}\| - \eta_n^2\varepsilon = \eta_{n+1}\eta_n\,\|P_n\widehat{x}\| + \eta_{n+1}^2\varepsilon \geq \eta_k\eta_n\,\|P_n\widehat{x}\| + \eta_k^2\varepsilon \geq \eta_k^2\,\|P_k\widehat{x}\| + \eta_k^2\varepsilon > \eta_k^2\,\|P_k\widehat{y}\|\,,$$

which implies that $|||y||| = \sup_{k\leq n}\eta_k^2\,\|P_k\widehat{y}\|$. Combining this with Lemma 32 we obtain that $|||\cdot|||$ is LFC-$\{e_i^*\}$.

Theorem 34 (Leung)**.** *Let M be a non-degenerate Orlicz function. There is a sequence $\{\eta_k\}$ of real numbers decreasing to 1 such that the norm on h_M defined by*

$$|||x||| = \sup_k \eta_k \|P_k \widehat{x}\|$$

is LFC-$\{e_i^*\}$ if and only if there is a $K > 1$ such that

$$\lim_{t\to 0+} \frac{M(Kt)}{M(t)} = \infty. \tag{3}$$

Proof. For the "if" part see the remark preceding the theorem. To show the "only if" part (which also appeared in [L1], but not precisely formulated and without proof), suppose that (3) does not hold and let $\{\eta_k\}$ be any sequence decreasing to 1. We will construct a vector $x \in S_{h_M}$ such that its coordinates form a positive non-increasing sequence and $\eta_k \|P_k x\| < 1$ for each $k \in \mathbb{N}$. Then obviously $|||x||| = 1$, but $|||P_n x||| = \max_{k\le n} \eta_k \|P_k x\| < 1$ for any $n \in \mathbb{N}$ and so $|||\cdot|||$ is not LFC-$\{e_i^*\}$ by Lemma 26.

Let $\{K_n\}$ be an increasing sequence of real numbers, $K_n > 1$ and $K_n \to \infty$. For each $n \in \mathbb{N}$ let $C_n > 2$ and $\{t_k^n\}_{k=1}^\infty$ be such that $\lim_{k\to\infty} t_k^n = 0$ and $M(K_n t_k^n) < C_n M(t_k^n)$ for all $k \in \mathbb{N}$. Let $\{\varepsilon_n\}$ be a sequence of real numbers such that $0 < \varepsilon_n < \frac{1}{2}$ and $\sum_{n=1}^\infty \varepsilon_n C_n < \infty$. Put $m_0 = 1$ and find $A > 0$ such that $M(1/A) = 1$ (which means $\|e_i\| = A$ for any $i \in \mathbb{N}$).

We choose $t_1 \in \{t_k^1\}$ this way: Define

$$m_1 = \min\left\{k : \eta_k \left\|\sum_{i=1}^k t_1 e_i\right\| \ge 1\right\},$$

and choose $t_1 \in \{t_k^1\}$ small enough such that

$$M(t_1) < \varepsilon_2 \quad \text{and} \tag{4}$$

$$\eta_{m_1} < 1 + \frac{\varepsilon_2}{1-\varepsilon_2}\frac{K_1 - 1}{C_1 - 1}. \tag{5}$$

By the convexity of M we have

$$\begin{aligned} M(\eta_{m_1} t_1) &\le \left(1 - \frac{\eta_{m_1}-1}{K_1-1}\right) M(t_1) + \frac{\eta_{m_1}-1}{K_1-1} M(K_1 t_1) < \left(1 - \frac{\eta_{m_1}-1}{K_1-1}\right) M(t_1) + \frac{\eta_{m_1}-1}{K_1-1} C_1 M(t_1) \\ &= \left(1 + (\eta_{m_1}-1)\frac{C_1-1}{K_1-1}\right) M(t_1) < \left(1 + \frac{\varepsilon_2}{1-\varepsilon_2}\right) M(t_1) = \frac{1}{1-\varepsilon_2} M(t_1), \end{aligned} \tag{6}$$

where the last inequality follows from (5). By the definition of m_1 we have $m_1 M(\eta_{m_1} t_1) \ge 1$. Consequently, using this inequality together with (6), $m_1 M(t_1) > m_1(1-\varepsilon_2) M(\eta_{m_1} t_1) \ge 1 - \varepsilon_2$. Hence, by (4),

$$(m_1 - 1) M(t_1) > 1 - 2\varepsilon_2.$$

We put $x_1 = \sum_{i=1}^{m_1-1} t_1 e_i$. Notice that by the definition of m_1 we have $1/\eta_{m_1-1} > \|x_1\| \ge 1/\eta_{m_1} - A t_1$.

Let us continue by induction. Fix any $j > 1$. Suppose we have $t_i \in \{t_k^i\}$, $m_i \in \mathbb{N}$ and $x_i \in h_M$ already defined for all $i < j$ such that $\sum_{k=1}^i (m_k - m_{k-1}) M(t_k) > 1 - 2\varepsilon_{i+1}$, $1/\eta_{m_i-1} > \|x_i\| \ge 1/\eta_{m_i} - A t_i$ and

$$x_i = \sum_{l=1}^{i} \sum_{k=m_{l-1}}^{m_l - 1} t_l e_k.$$

We choose $t_j \in \{t_k^j\}$ this way: Define

$$m_j = \min\left\{k \ge m_{j-1} : \eta_k \left\| x_{j-1} + \sum_{i=m_{j-1}}^k t_j e_i \right\| \ge 1\right\},$$

and choose $t_j \in \{t_k^j\}$ small enough such that

$$M(t_j) < \varepsilon_{j+1} \quad \text{and} \tag{7}$$

$$\eta_{m_j} < 1 + \frac{\varepsilon_{j+1}}{1-\varepsilon_{j+1}} \min_{1\le i\le j}\left\{\frac{K_i - 1}{C_i - 1}\right\}. \tag{8}$$

Notice that this is possible since $\|x_{j-1}\| < 1/\eta_{m_{j-1}-1}$. Using again the convexity of M, the fact that $t_i \in \{t_k^i\}$ and (8), for any $1 \le i \le j$ we obtain

$$\begin{aligned} M(\eta_{m_j} t_i) &\le \left(1 - \frac{\eta_{m_j}-1}{K_i-1}\right) M(t_i) + \frac{\eta_{m_j}-1}{K_i-1} M(K_i t_i) < \left(1 - \frac{\eta_{m_j}-1}{K_i-1}\right) M(t_i) + \frac{\eta_{m_j}-1}{K_i-1} C_i M(t_i) \\ &= \left(1 + (\eta_{m_j}-1)\frac{C_i-1}{K_i-1}\right) M(t_i) < \left(1 + \frac{\varepsilon_{j+1}}{1-\varepsilon_{j+1}}\right) M(t_i) = \frac{1}{1-\varepsilon_{j+1}} M(t_i). \end{aligned}$$

These estimates together with the definition of m_j and x_{j-1} give

$$\sum_{i=1}^{j-1}(m_i - m_{i-1})M(t_i) + (m_j - m_{j-1} + 1)M(t_j)$$
$$> (1-\varepsilon_{j+1})\left(\sum_{i=1}^{j-1}(m_i - m_{i-1})M(\eta_{m_j} t_i) + (m_j - m_{j-1} + 1)M(\eta_{m_j} t_j)\right) \geq 1 - \varepsilon_{j+1},$$

so the use of (7) yields

$$\sum_{i=1}^{j}(m_i - m_{i-1})M(t_i) > 1 - 2\varepsilon_{j+1}. \tag{9}$$

We put

$$x_j = \sum_{i=1}^{j}\sum_{k=m_{i-1}}^{m_i - 1} t_i e_k$$

and notice that, by the definition of m_j,

$$1/\eta_{m_j - 1} > \|x_j\| \geq 1/\eta_{m_j} - At_j. \tag{10}$$

We have inductively constructed a sequence $\{x_j\} \subset h_M$ given by the formula above, such that $\|x_j\| < 1$ and (9) holds for any $j \in \mathbb{N}$. Choose any $j > 1$. Since $\|x_j\| < 1$, it follows that $\sum_{i=1}^{j}(m_i - m_{i-1})M(t_i) < 1$ and comparing this with (9) for $j-1$ we obtain

$$(m_j - m_{j-1})M(t_j) < 2\varepsilon_j.$$

This implies that $x_j \to x \in h_M$. Indeed, suppose $K > 0$. Let $n \in \mathbb{N}$ be such that $K_n \geq K$. Then

$$\sum_{i=n}^{\infty}(m_i - m_{i-1})M(Kt_i) \leq \sum_{i=n}^{\infty}(m_i - m_{i-1})M(K_i t_i) \leq \sum_{i=n}^{\infty}(m_i - m_{i-1})C_i M(t_i) < 2\sum_{i=n}^{\infty}\varepsilon_i C_i < \infty$$

and so by the basic properties of h_M the vector $x = \sum_{i=1}^{\infty}\sum_{k=m_{i-1}}^{m_i - 1} t_i e_k$ belongs to h_M. This means also that $t_j \to 0$ and thus from (10) we can conclude that $\|x\| = \lim\|x_j\| = 1$. Moreover, the construction of x_j (namely the choice of m_j) guarantees that $\eta_k \|P_k x\| < 1$ for each $k \in \mathbb{N}$.

□

The following theorem is a strengthening of a theorem from [L1]. Leung's statement is that the Orlicz sequence space h_M does not admit a LFC norm if M satisfies the condition below.

Theorem 35. *Let M be a non-degenerate Orlicz function for which there exists a sequence $\{t_n\}$ decreasing to 0 such that*

$$\sup_n \frac{M(Kt_n)}{M(t_n)} < \infty \quad \textit{for all } 0 < K < \infty.$$

Then the Orlicz sequence space h_M is not locally flat.

Proof. Suppose that there is a non-empty bounded $A \subset h_M$ which is LFD. Without loss of generality we may assume that $0 \in A \subset B_X$ and A is LFD-$\mathcal{Z}$, where $\mathcal{Z} = \{\overline{\operatorname{span}}\{e_i\}_{i=n}^{\infty};\ n \in \mathbb{N}\}$. (Since h_M is c_0-saturated by Theorem 16, it does not contain ℓ_1. As $\{e_i\}$ is unconditional, it is shrinking by James's theorem. Now consider $T(\overline{A})$, where $T: X \to X$ is an equivalence isomorphism of the bases $\{x_i\}$ and $\{e_i\}$ from Corollary 25.)

Notice, that the vectors with coordinates in the set $\{t_n\} \cup \{0\}$ have the property of "bounded completeness": If we have $\left\|\sum_{i=1}^{k} t_{m_i} e_i\right\| \leq 1$ for all $k \in \mathbb{N}$, where $m_i \in \mathbb{N} \cup \{0\}$ are not necessarily distinct (we put $t_0 = 0$), then $\sum_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M. Indeed, it follows that $\sum_{i=1}^{k} M(t_{m_i}) \leq 1$ for all $k \in \mathbb{N}$. For all $0 < K < \infty$ and all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} M(Kt_{m_i}) \leq \sup_n \frac{M(Kt_n)}{M(t_n)} \sum_{i=1}^{k} M(t_{m_i}) \leq \sup_n \frac{M(Kt_n)}{M(t_n)}.$$

Consequently, $\sum_{i=1}^{\infty} M(Kt_{m_i}) < \infty$ for all $0 < K < \infty$, and the sum $\sum_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M.

We construct a sequence $\{x_k\} \subset A$ by induction. Put $x_0 = 0 \in A$ and define natural numbers $m_0 = n_0 = 1$. If $m_{k-1} \in \mathbb{N}$, $n_{k-1} \in \mathbb{N}$ and $x_{k-1} \in A$ are already defined, we put

$$M_k = \{(m, n) \in \mathbb{N}^2 \colon m \geq m_{k-1}, n > n_{k-1} \text{ and } x_{k-1} + t_m e_n \in A\}.$$

As A is determined by some $W \in \mathcal{Z}$ on a neighbourhood of x_{k-1}, where W contains all e_n for n big enough, and $t_m \to 0$, we can see that $M_k \neq \emptyset$. Let $(m_k, n_k) = \min M_k$ in the lexicographic ordering of $\mathbb{N}^2$ and put $x_k = x_{k-1} + t_{m_k} e_{n_k}$.

Since $\{x_k\} \subset A \subset B_X$ and $x_k = \sum_{i=1}^{k} t_{m_i} e_{n_i}$, by the above argument $x_k \to x \in h_M$. We can find $\delta > 0$ so that A is determined by some $Z \in \mathcal{Z}$ on $U(x, \delta)$. There is $N \in \mathbb{N}$ such that $\{e_i\}_{i>N} \subset Z$. Because x_k converges, we have $m_k \to \infty$ and so there is $j \in \mathbb{N}$ such that $x_j \in U(x, \delta/2)$, $\|t_{m_j} e_1\| < \delta/2$, $m_j < m_{j+1}$ and $n_j > N$. Then $x_j + t_{m_j} e_{n_j+1} \in A$ and therefore $(m_j, n_j + 1) \in M_{j+1}$. But $(m_j, n_j + 1) < (m_{j+1}, n_{j+1})$, which is a contradiction.

□

In [L1], Leung constructed a c_0-saturated Orlicz sequence space satisfying the condition in Theorem 35. Therefore, we have the following corollary:

Corollary 36. *Leung's space is a separable c_0-saturated Asplund space that is not locally flat.*

The main construction of this paper is contained in the next theorem.

Theorem 37. *Let M be a non-degenerate Orlicz function for which there exist sequence $F_k \subset (0,1]$ such that*

(i) $\lim_{k\to\infty}(\sup F_k) = 0$,
(ii) there is a sequence $K_k > 1$ such that

$$\lim_{\substack{t\to 0+\\ t\notin F_k}} \frac{M(K_k t)}{M(t)} = \infty,$$

(iii) there is a $K > 1$ and a sequence $C_k \to \infty$ such that $M(Kt) \geq C_k M(t)$ for all $t \in F_k$.

Then there exists a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump function on the Orlicz sequence space h_M.

Proof. Without loss of generality we may and do assume that $M(1) = 1$ (i.e. $\|e_1\| = 1$) and $C_k \geq C_1 > 0$ for any $k \in \mathbb{N}$.

For each $t \in \overline{F_k} \setminus \{0\}$ choose $\varepsilon_t^k > 0$ such that $M(s) < 2M(t)$ and $t/2 < s < 2t$ if $|s - t| < \varepsilon_t^k$. Let us define sets $G_k = \bigcup_{t\in\overline{F_k}\setminus\{0\}}(t - \varepsilon_t^k, t + \varepsilon_t^k)$. Then each G_k is open, $G_k \supset (\overline{F_k} \setminus \{0\})$ and $\sup G_k \leq 2\sup F_k$. Moreover, for any $s \in G_k$ the choice of an appropriate $t \in \overline{F_k} \setminus \{0\}$ from the definition of G_k yields $M(2Ks) > M(Kt) \geq C_k M(t) > C_k M(s)/2$ (using (iii) and the continuity of M). So, if we multiply K by 2 and each C_k by $\frac{1}{2}$ and denote these new constants K and C_k again to avoid carrying unnecessary factors, we have

$$\lim_{k\to\infty}(\sup G_k) = 0, \tag{11}$$

$$M(Kt) \geq C_k M(t) \quad \text{for all } t \in G_k. \tag{12}$$

Let us define a sequence of continuous functions φ_k on $[0, +\infty)$ such that $0 \leq \varphi_k(t) \leq t$, $\varphi_k(t) = 0$ for $t \in F_k$ and $\varphi_k(t) = t$ for $t \notin G_k$, and a mapping $\phi_k\colon h_M \to h_M$ by $\phi_k(x) = \big(\varphi_k(|x_i|)\big)$ for $x = (x_i) \in h_M$. (We can take for example $\varphi_k(t) = t\,\mathrm{dist}(t, F_k)/\big(\mathrm{dist}(t, F_k) + \mathrm{dist}(t, \mathbb{R} \setminus G_k)\big)$ for $t > 0$ and $\varphi_k(0) = 0$.)

Fix $k \in \mathbb{N}$.

First, observe that the mapping $\phi_k\colon h_M \to h_M$ is continuous: Choose $x \in h_M$ and $\varepsilon > 0$ and find $n \in \mathbb{N}$ such that $\|R_n x\| < \frac{\varepsilon}{8}$. As φ_k is continuous, there is $\delta > 0$ such that $\big||x_i| - |y_i|\big| < \delta$ implies $\big|\varphi_k(|x_i|) - \varphi_k(|y_i|)\big| < \frac{\varepsilon}{2n}$ for all $1 \leq i \leq n$. We have $\big||x_i| - |y_i|\big| \leq |x_i - y_i| = \|(x - y)_i e_i\| \leq \|x - y\|$. (The last inequality uses the fact that the norm $\|\cdot\|$ is a lattice norm.) Thus, whenever $\|x - y\| < \min\{\delta, \frac{\varepsilon}{4}\}$,

$$\begin{aligned}
\|\phi_k(x) - \phi_k(y)\| &\leq \|P_n(\phi_k(x) - \phi_k(y))\| + \|R_n(\phi_k(x) - \phi_k(y))\| \\
&\leq \sum_{i=1}^n \big|\varphi_k(|x_i|) - \varphi_k(|y_i|)\big| + \|R_n\phi_k(x)\| + \|R_n\phi_k(y)\| < \frac{\varepsilon}{2} + \|R_n x\| + \|R_n y\| \\
&\leq \frac{\varepsilon}{2} + \|R_n x\| + \|R_n x\| + \|R_n(x - y)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

The third and the fifth inequality follow again from the fact that the norm $\|\cdot\|$ is lattice.

Claim 1. *There is a non-increasing sequence $\{\eta_n^k\} \subset \mathbb{R}$ satisfying $\eta_n^k \leq 2$ and $\lim_{n\to\infty}\eta_n^k = 1$, such that for each $x \in h_M$ for which $\phi_k(x) \neq 0$ there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for any $y \in U(x, \delta)$ we have*

$$\eta_n^k \nu\big(P_n\widehat{\phi_k(y)}\big) > \nu\big(\widehat{\phi_k(y)}\big) \quad \textit{for all } n \geq n_0.$$

We will construct the sequence η_n^k as follows: If $(0, a) \subset F_k$ for some $a > 0$, then any non-increasing sequence $\eta_n^k \to 1$ such that $1 < \eta_n^k \leq 2$ for all $n \in \mathbb{N}$ will do. Indeed, then there is $n_0 \in \mathbb{N}$ such that $|x_i| < a/2$ for $i \geq n_0$ and hence $\widehat{\phi_k(y)} = P_{n_0}\widehat{\phi_k(y)}$ whenever $\|x - y\| < a/2$.

Otherwise, put $b_n = \inf\Big\{\frac{M(K_k t)}{M(t)};\ 0 < t \leq \sqrt{M^{-1}(\frac{1}{n})},\ t \notin F_k\Big\}$. By our assumption, $b_n < \infty$ for all $n \in \mathbb{N}$. Notice, that b_n is non-decreasing and, by (ii), $b_n \to \infty$. Define $\eta_n^k = \min\big\{2, (1 - b_n^{-1/2})^{-1}\big\}$. It is trivial to check that η_n^k is non-increasing and $\eta_n^k \to 1$.

Define a mapping $Q_k\colon h_M \to h_M$ by $Q_k(x)_i = |x_i|$ if $|x_i| \notin F_k$, $Q_k(x)_i = 0$ otherwise.

Now choose $x \in h_M$ for which $\phi_k(x) \neq 0$. By Lemma 33 there is $0 < \delta < \frac{1}{2K_k}$ such that $\nu(\phi_k(y)) > \frac{1}{2}\nu(\phi_k(x))$ if $\|x - y\| < \delta$. Find $n_0 \in \mathbb{N}$ such that $\eta_n^k = (1 - b_n^{-1/2})^{-1}$, $b_n^{-1/2} < \frac{1}{2}\nu(\phi_k(x))$, $\|R_n x\| < \frac{1}{2K_k}$ and $M^{-1}(\frac{1}{n}) \leq 1/(\|x\| + \delta)^2$

for $n \geq n_0$. Fix $n \geq n_0$ and $y \in h_M$ such that $\|x - y\| < \delta$. Using Lemma 31(b) and the fact that the canonical norm on h_M is a symmetric lattice norm, we have

$$\left\| R_n \widehat{Q_k(y)} \right\| \leq \|R_n Q_k(y)\| \leq \|R_n y\| \leq \|R_n x\| + \|R_n(x-y)\| < \frac{1}{K_k}. \tag{13}$$

As $\sum_{i=1}^{\infty} M\big(\widehat{Q_k(y)}_i / \|y\|\big) \leq \sum_{i=1}^{\infty} M(|y_i| / \|y\|) = \nu(y/\|y\|) = 1$ and the sequence $\widehat{Q_k(y)}_i$ is non-increasing, it follows that $\widehat{Q_k(y)}_i / \|y\| \leq M^{-1}(\frac{1}{i})$ for any $i \in \mathbb{N}$. From the definition of n_0 we obtain $\widehat{Q_k(y)}_i \leq \|y\| \, M^{-1}(\frac{1}{i}) \leq (\|x\|+\delta) M^{-1}(\frac{1}{i}) \leq \sqrt{M^{-1}(\frac{1}{i})}$ for $i \geq n_0$. Notice further that $\widehat{Q_k(y)}_i \notin F_k$ for any $i \in \mathbb{N}$, thus by the definition of b_n and (13) we have

$$1 > \nu\big(K_k R_n \widehat{Q_k(y)}\big) = \sum_{i>n} M\big(K_k \widehat{Q_k(y)}_i\big) \geq \sum_{i>n} b_i M\big(\widehat{Q_k(y)}_i\big) \geq b_n \sum_{i>n} M\big(\widehat{Q_k(y)}_i\big),$$

which together with the easily checked inequality $\widehat{\phi_k(y)}_i \leq \widehat{Q_k(y)}_i$ for any $i \in \mathbb{N}$ implies

$$\sum_{i>n} M\big(\widehat{\phi_k(y)}_i\big) \leq \sum_{i>n} M\big(\widehat{Q_k(y)}_i\big) \leq \frac{1}{b_n}.$$

Notice that by the definition of δ and n_0 and by the symmetry of ν we have $\nu(\widehat{\phi_k(y)}) > b_n^{-1/2}$ and therefore (use this fact for the second inequality)

$$\nu\big(P_n \widehat{\phi_k(y)}\big) = \sum_{i=1}^{n} M\big(\widehat{\phi_k(y)}_i\big) \geq \sum_{i=1}^{\infty} M\big(\widehat{\phi_k(y)}_i\big) - \frac{1}{b_n} = \nu\big(\widehat{\phi_k(y)}\big) - \frac{1}{b_n} > (1 - b_n^{-1/2}) \nu\big(\widehat{\phi_k(y)}\big) = \frac{1}{\eta_n^k} \nu\big(\widehat{\phi_k(y)}\big),$$

which proves the claim.

Choose an arbitrary sequence $\{\gamma_k\} \subset \mathbb{R}$ decreasing to 1. Let us define a sequence of functions $g_k \colon h_M \to \mathbb{R}$ by

$$g_k(x) = \frac{1}{C_k} + \sup_n \gamma_{k+n} \eta_n^k \nu\big(P_n \widehat{\phi_k(x)}\big).$$

Claim 2. *Each g_k is a LFC-$\{e_i^*\}$ function on $\{x \in h_M,\ \phi_k(x) \neq 0\}$ and continuous on h_M.*

Indeed, for a fixed $k \in \mathbb{N}$ and $x \in h_M$, $\phi_k(x) \neq 0$, choose an appropriate δ and n_0 from Claim 1. Let $N \geq n_0$ be such that $\gamma_{k+n} \eta_n^k < \gamma_{k+n_0}$ whenever $n > N$. Then for $y \in U(x, \delta)$ and $n > N$ we have

$$\gamma_{k+n_0} \eta_{n_0}^k \nu\big(P_{n_0} \widehat{\phi_k(y)}\big) > \gamma_{k+n} \eta_n^k \nu\big(\widehat{\phi_k(y)}\big) \geq \gamma_{k+n} \eta_n^k \nu\big(P_n \widehat{\phi_k(y)}\big)$$

and hence

$$g_k(y) = \frac{1}{C_k} + \max_{1 \leq n \leq N} \gamma_{k+n} \eta_n^k \nu\big(P_n \widehat{\phi_k(y)}\big). \tag{14}$$

By Lemma 32 there is a neighbourhood V of $\phi_k(x)$ and a finite $A \subset \mathbb{N}$ such that the function $\max_{1 \leq n \leq N} \gamma_{k+n} \eta_n^k \nu(P_n \widehat{z})$ depends only on $\{e_i^*\}_{i \in A}$ on V. But since ϕ_k is continuous, there is a neighbourhood U of x, $U \subset U(x, \delta)$, such that $\phi_k(U) \subset V$. Further, as $\phi_k(y)_i = \phi_k(z)_i$ whenever $y_i = z_i$ for any $i \in \mathbb{N}$, the function g_k depends only on $\{e_i^*\}_{i \in A}$ on U.

Moreover, each g_k is continuous on h_M: Using the continuity of ϕ_k, Lemma 31(d) and (14) we can see that g_k is continuous on $\{x \in h_M,\ \phi_k(x) \neq 0\}$. On the other hand,

$$\frac{1}{C_k} \leq g_k(x) \leq \frac{1}{C_k} + \gamma_k \eta_1^k \nu\big(\phi_k(x)\big),$$

and the continuity of g_k at any x with $\phi_k(x) = 0$ follows from the continuity of ϕ_k and the properties of ν.

Notice further that, since ν is lattice,

$$g_k(x) \leq \frac{1}{C_k} + \gamma_k \eta_1^k \nu(x), \tag{15}$$

and as $g_k(x) \geq \frac{1}{C_k} + \gamma_{k+n} \eta_n^k \nu\big(P_n \widehat{\Phi_k(x)}\big)$ for each $n \in \mathbb{N}$, the continuity of ν implies

$$g_k(x) \geq \frac{1}{C_k} + \nu\big(\phi_k(x)\big), \tag{16}$$

for any $x \in h_M$ and any $k \in \mathbb{N}$.

Claim 3. *For each $x \in h_M$ there is $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for any $y \in U(x, \delta)$ and $k \geq k_0$ we have*

$$\nu(y) < \frac{1}{C_k} + \nu\big(\phi_k(y)\big).$$

Indeed, choose an arbitrary $x \in h_M$. Let $n \in \mathbb{N}$ be such that $\|R_n x\| < \frac{1}{3K}$ and $0 < \delta < \frac{1}{3K}$ be such that moreover $\delta \leq \frac{1}{2}\min\{|x_i|;\ x_i \neq 0, i \leq n\}$ if $P_n x \neq 0$. Pick any $y \in h_M$ for which $\|x - y\| < \delta$. Notice that if $|y_i| < \delta$ then either $x_i = 0$ or $i > n$. Let $A_1 = \{i;\ x_i = 0\}$, $A_2 = \{i;\ i > n\}$. Then

$$\|P_{A_1 \cup A_2} y\| \leq \|P_{A_1} y\| + \|R_n y\| \leq \|P_{A_1}(y - x)\| + \|R_n x\| + \|R_n(x - y)\| < \frac{1}{K}.$$

Therefore we have $\sum_{|y_i|<\delta} M\big(K|y_i|\big) < 1$. By (11) we can find $k_0 \in \mathbb{N}$ such that $G_k \subset (0, \delta)$ for all $k \geq k_0$ and hence, using (12),

$$\sum_{|y_i| \in G_k} M\big(|y_i|\big) < \frac{1}{C_k} \quad \text{for all } k \geq k_0.$$

It follows that, for any $y \in U(x, \delta)$ and $k \geq k_0$,

$$\begin{aligned}\nu(y) = \sum_{i=1}^{\infty} M\big(|y_i|\big) &= \sum_{|y_i| \in G_k} M\big(|y_i|\big) + \sum_{|y_i| \notin G_k} M\big(|y_i|\big) \\ &= \sum_{|y_i| \in G_k} M\big(|y_i|\big) + \sum_{|y_i| \notin G_k} M\big(\phi_k(y)_i\big) < \frac{1}{C_k} + \sum_{i=1}^{\infty} M\big(\phi_k(y)_i\big) = \frac{1}{C_k} + \nu\big(\phi_k(y)\big).\end{aligned}$$

Finally let us define a function $g\colon h_M \to \mathbb{R}$ by

$$g(x) = \sup_k \gamma_k g_k(x).$$

Choose $0 \neq x \in h_M$ and find δ and k_0 from Claim 3. Since ν is continuous, we may also assume that $\nu(y) \geq \nu(x)/2$ if $\|x - y\| < \delta$. There is $N \in \mathbb{N}$ such that $2\gamma_k/(\nu(x)C_k) + \gamma_k^2 \eta_1^k < \gamma_{k_0}$ for $k > N$. Then for any $y \in U(x, \delta)$ and $k > N$ we have (using first (15), then the definition of N, Claim 3 and finally (16))

$$\gamma_k g_k(y) \leq \frac{\gamma_k}{C_k} + \gamma_k^2 \eta_1^k \nu(y) < \gamma_{k_0} \nu(y) < \frac{\gamma_{k_0}}{C_{k_0}} + \gamma_{k_0} \nu\big(\phi_{k_0}(y)\big) \leq \gamma_{k_0} g_{k_0}(y). \tag{17}$$

This means that

$$g(y) = \sup_k \gamma_k g_k(y) = \max_{k \leq N} \gamma_k g_k(y) \tag{18}$$

for $y \in U(x, \delta)$. In particular, since each g_k is continuous on h_M, it follows that g is continuous on $h_M \setminus \{0\}$. On the other hand, for any $y \in h_M$,

$$\frac{\gamma_1}{C_1} \leq \gamma_1 g_1(y) \leq g(y) \leq \frac{\gamma_1}{C_1} + 2\gamma_1^2 \nu(y),$$

(the last inequality follows from (15)) and the continuity of ν implies that g is continuous at 0 and hence on the whole of h_M.

Let us define a set $D = \{x \in h_M,\ g(x) > \frac{\gamma_1}{C_1}\}$. Choose any $x \in D$ and find an appropriate N and δ for this x as above. Let $A = \{k\colon 1 \leq k \leq N,\ \phi_k(x) \neq 0\}$. If $k \in \{1, \dots, N\} \setminus A$, then

$$\gamma_k g_k(x) = \frac{\gamma_k}{C_k} \leq \frac{\gamma_1}{C_1} < g(x).$$

By the continuity of all ϕ_k, g_k and g, there is a neighbourhood U of x, $U \subset U(x, \delta)$, such that $\phi_k(y) \neq 0$ for $k \in A$ and $\gamma_k g_k(y) < g(y)$ for $k \in \{1, \dots, N\} \setminus A$ whenever $y \in U$. Thus, by (18), $g(y) = \max_{k \in A} \gamma_k g_k(y)$ for $y \in U$. Since each g_k, $k \in A$, is LFC on U, so is g. Therefore g is LFC on D.

From the last two inequalities in (17) we can see that $g(x) > \nu(x)$ for any $x \in h_M$. Thus $g(x) > \|x\|$ on $\{x \in h_M;\ \|x\| \geq 1\}$ and we can compose g with a suitable real continuous function to obtain a desired continuous LFC bump. To finish the proof, it remains to apply Theorem 30.

□

Theorem 38. *There is a non-degenerate Orlicz function M such that $\liminf_{t \to 0+} \frac{M(Kt)}{M(t)} < \infty$ for any $K > 1$, yet the corresponding Orlicz sequence space h_M admits a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump.*

Proof. Suppose we have a sequence $b_n \geq 1$, $n \geq 0$. For $n = 0, 1, 2, \dots$, put $a_n = \prod_{m=0}^{n} b_m^{-1}$ and let $M(t)$ be a piecewise linear continuous function on $[0, \infty)$, such that $M(0) = 0$, $M'(t) = a_n$ for $2^{-(n+1)} < t < 2^{-n}$ and $M'(t) = 1$ for $t > 1$. Clearly, this is a non-degenerate Orlicz function and the constants b_n determine the ratio of the slopes of M on the two consecutive dyadic intervals. Suppose that $j \in \mathbb{N} \cup \{0\}$ and $2^{-(n+1)} \leq t \leq 2^{-n}$ for some $n \geq j$. Then

$$2^{j-n-2} a_{n-j+1} \leq M(2^{j-n-1}) \leq M(2^j t) \leq M(2^{j-n}) \leq 2^{j-n} a_{n-j}.$$

Hence, for $n \geq j \geq 2$,

$$2^{j-2} \prod_{m=n-j+2}^{n} b_m \leq \frac{M(2^j t)}{M(t)} \leq 2^{j+2} \prod_{m=n-j+1}^{n+1} b_m. \tag{19}$$

If F_k is chosen to be $\bigcup_{n \in I_k} [2^{-(n+1)}, 2^{-n})$ for some $I_k \subset \mathbb{N}$, then for conditions (i) to (iii) in Theorem 37 to hold, it is sufficient to require

(a) $\lim\limits_{k\to\infty} \min I_k = \infty$,

(b) For each $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$ such that $\lim\limits_{\substack{n\to\infty \\ n\notin I_k}} \max\{b_{n-j_k}, \dots, b_n\} = \infty$,

(c) $\lim\limits_{k\to\infty} \min\limits_{n\in I_k} b_n = \infty$.

Indeed, (a) implies (i). If $t \in (0,1) \setminus F_k$, then there is $n \notin I_k$ such that $t \in [2^{-(n+1)}, 2^{-n})$ and thus (19) together with (b) implies (ii) for $K_k = 2^{j_k+2}$. Finally, (19) together with (c) implies (iii) for $K = 4$ and $C_k = \min_{n\in I_k} b_n$. On the other hand, condition

(d) $\liminf\limits_{n\to\infty} \max\{b_{n-j}, \dots, b_n\} < \infty$ for all $j \in \mathbb{N}$

with (19) ensures that $\liminf_{t\to 0+} \frac{M(Kt)}{M(t)} < \infty$ for any $K > 1$.

Now we construct a sequence $b_n \geq 1$, $n \geq 0$ and a sequence $I_k \subset \mathbb{N}$ satisfying conditions (a) to (d). Choose a non-decreasing sequence $\{c_n\} \subset \mathbb{R}$ such that $c_n \geq 1$ and $c_n \to \infty$. For $i = 0, 1, 2, \dots$, $j = 0, \dots, i$ and $k = 0, \dots, j+1$, let

$$n(i,j,k) = \sum_{l=0}^{i-1} \sum_{m=1}^{l+1} (m+1) + \sum_{m=1}^{j} (m+1) + k$$

and define $\{b_n\}_{n=0}^{\infty}$ by $b_{n(i,j,0)} = c_i$ and $b_{n(i,j,k)} = c_j$ for $k = 1, \dots, j+1$. The sequence $\{b_n\}$ fills a triangular table, where the index $n = n(i,j,k)$ is interpreted as follows: i counts the rows, by j we index groups of columns, where the jth group consists of $j+2$ columns, and k is an index of a column in the jth group. So we have the following table

$$\begin{array}{llllllllllllll}
b_0 & b_1 \\
b_2 & b_3 & b_4 & b_5 & b_6 \\
b_7 & b_8 & b_9 & b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\
b_{16} & b_{17} & b_{18} & b_{19} & b_{20} & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} & b_{29} \\
\dots & \dots
\end{array}$$

with the values

$$\begin{array}{llllllllllllll}
c_0 & c_0 \\
c_1 & c_0 & c_1 & c_1 & c_1 \\
c_2 & c_0 & c_2 & c_1 & c_1 & c_2 & c_2 & c_2 & c_2 \\
c_3 & c_0 & c_3 & c_1 & c_1 & c_3 & c_2 & c_2 & c_2 & c_3 & c_3 & c_3 & c_3 & c_3 \\
\dots & \dots
\end{array}$$

For any $j \in \mathbb{N}$ we have $\max\{b_{n(i,j,1)}, \dots, b_{n(i,j,j+1)}\} = c_j$ for all $i \geq j$ and (d) is clearly satisfied.

Now let $I_k = \bigcup\limits_{m=k-1}^{\infty} \bigcup\limits_{i=m}^{\infty} \{n(i,m,1), \dots, n(i,m,m+1)\}$ for $k \in \mathbb{N}$, i.e. I_k consists of all the columns in the table starting with the $(k-1)$th group but without the first column in each group. Condition (a) obviously holds. If $n(i,j,l) \notin I_k$, then $l \leq j+1 < k$ or $l = 0$ but in both cases $\max\{b_{n(i,j,l)-k+1}, \dots, b_{n(i,j,l)}\} \geq b_{n(i,j,0)} = c_i$ and hence (b) is satisfied. Finally, $\min_{n\in I_k} b_n = c_{k-1}$ implies (c).

$\square$

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