APPROXIMATION OF LIPSCHITZ MAPPINGS

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ABSTRACT. We prove that any Lipschitz mapping from a separable Banach space into any Banach space can be approximated by uniformly Gâteaux differentiable Lipschitz mapping.

If X is a separable Banach space and Y is a Banach space with RNP, then any Lipschitz mapping from X to Y can be approximated by a Gâteaux differentiable Lipschitz mapping ([BS]; cf. [BL, page 155]). The aim of this paper is to show that using a different technique of the proof the assumption on the target space having RNP can be dropped and moreover the approximation can be made uniformly Gâteaux differentiable.

Let X, Y be Banach spaces, f a mapping $f: X \to Y$. Let us define the directional derivative of f at $x \in X$ in the direction $h \in X$ as $D_h f(x) = \lim_{t\to 0} \frac{1}{t}(f(x+th) - f(x))$. If for any fixed x the directional derivative exists for all $h \in X$ and $D_h f(x)$ is a bounded linear operator in h, we say that f is Gâteaux differentiable at x. If moreover for all fixed h the limit defining $D_h f(x)$ is uniform for $x \in X$ we say that f is uniformly Gâteaux differentiable (UG for short). For any other unexplained term we refer to [BL].

Theorem. *Let X be a separable Banach space, Y a Banach space, $f: X \to Y$ be an L-Lipschitz mapping and $\varepsilon > 0$. Then there is a mapping $g: X \to Y$ which is L-Lipschitz, UG and $\|f - g\| < \varepsilon$.*

Proof. We will construct the function g by using the method of integral convolution in a countable set of directions which was presented in [FWZ].

Let $\{h_i\}$ be a dense subset of S_X and $\varphi_i: \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \dots$ be such that $\varphi_i \geq 0$, $\varphi_i \in C^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi_i = 1$ and $\operatorname{supp} \varphi_i \subset \left[-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i}\right]$.

Define a mapping $g_n: X \to Y$, $n = 1, 2, \dots$ by

$$g_n(x) = \int_{\mathbb{R}^n} f\left(x - \sum_{i=1}^n t_i h_i\right) \prod_{i=1}^n \varphi_i(t_i) \, \mathrm{d}\lambda_n(t),$$

where we integrate in the Bochner sense with respect to the n-dimensional Lebesgue measure.

The mappings g_n are L-Lipschitz:

$$\|g_n(x) - g_n(y)\| \leq \int_{\mathbb{R}^n} \left\| f\left(x - \sum_{i=1}^n t_i h_i\right) - f\left(y - \sum_{i=1}^n t_i h_i\right) \right\| \prod_{i=1}^n \varphi_i(t_i) \, \mathrm{d}\lambda_n \leq L \|x - y\| \int_{\mathbb{R}^n} \prod_{i=1}^n \varphi_i(t_i) \, \mathrm{d}\lambda_n = L \|x - y\|.$$

There is a mapping g such that $g_n \to g$ uniformly on X (and hence g is also L-Lipschitz). Indeed, denote by M_m the set $\prod_{i=1}^m \left[-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i}\right] \subset \mathbb{R}^m$. Then using the Fubini theorem and the fact that $\int_{\mathbb{R}} \varphi_i = 1$ for any i we have for $m > n$ and any $x \in X$

$$\begin{aligned} \|g_m(x) - g_n(x)\| &= \left\| \int_{\mathbb{R}^m} \left(f\left(x - \sum_{i=1}^m t_i h_i\right) - f\left(x - \sum_{i=1}^n t_i h_i\right) \right) \prod_{i=1}^m \varphi_i(t_i) \, \mathrm{d}\lambda_m \right\| \leq L \int_{\mathbb{R}^m} \left\| \sum_{i=n+1}^m t_i h_i \right\| \prod_{i=1}^m \varphi_i(t_i) \, \mathrm{d}\lambda_m \\ &\leq L \int_{M_m} \left(\sum_{i=n+1}^m |t_i| \right) \prod_{i=1}^m \varphi_i(t_i) \, \mathrm{d}\lambda_m \leq L \left(\sum_{i=n+1}^m \frac{\varepsilon}{2L2^i} \right) \int_{M_m} \prod_{i=1}^m \varphi_i(t_i) \, \mathrm{d}\lambda_m \leq \frac{\varepsilon}{2 \cdot 2^n}. \end{aligned}$$

Moreover, $\|f - g\| < \varepsilon$. Pick g_n such that $\|g_n - g\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} \|f(x) - g(x)\| &\leq \|f(x) - g_n(x)\| + \|g_n(x) - g(x)\| < \int_{\mathbb{R}^n} \left\| f(x) - f\left(x - \sum_{i=1}^n t_i h_i\right) \right\| \prod_{i=1}^n \varphi_i(t_i) \, \mathrm{d}\lambda_n + \frac{\varepsilon}{2} \\ &\leq L \int_{M_n} \left(\sum_{i=1}^n |t_i| \right) \prod_{i=1}^n \varphi_i(t_i) \, \mathrm{d}\lambda_n + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

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Now it remains to show that g is UG.

First we will show that the directional derivative $D_{h_i} g_n(x)$ exists for any $x \in X$ and $i = 1, 2, \dots, n$.

$$\begin{aligned}
D_{h_i} g_n(x) &= \lim_{\tau\to 0} \frac{1}{\tau}\big(g_n(x+\tau h_i) - g_n(x)\big) \\
&= \lim_{\tau\to 0} \frac{1}{\tau}\left(\int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j + \tau h_i\right)\prod_{j=1}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n - \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\prod_{j=1}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n\right) \\
&= \lim_{\tau\to 0} \frac{1}{\tau}\left(\int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\varphi_i(t_i+\tau)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n - \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\prod_{j=1}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n\right) \\
&= \lim_{\tau\to 0}\int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\frac{\varphi_i(t_i+\tau)-\varphi_i(t_i)}{\tau}\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n = \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\varphi_i'(t_i)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n
\end{aligned} \tag{1}$$

For the third equality we use the substitution $t_i \to t_i + \tau$. In order to show the last equality, choose $\eta > 0$. Then there is $0 < \delta \le 1$ such that $\left|\frac{1}{\tau}(\varphi_i(t_i+\tau) - \varphi_i(t_i)) - \varphi_i'(t_i)\right| < \eta$ for any $0 < |\tau| < \delta$ and $t_i \in \mathbb{R}$. (Use the Mean Value Theorem and the compactness of the support of φ_i.) Hence, for $0 < |\tau| < \delta$,

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\frac{\varphi_i(t_i+\tau)-\varphi_i(t_i)}{\tau}\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n - \int_{\mathbb{R}^n} f\left(x - \sum_{j=1}^n t_j h_j\right)\varphi_i'(t_i)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n\right\| \\
&\qquad\le \int_M \left\|f\left(x - \sum_{j=1}^n t_j h_j\right)\right\|\left|\frac{\varphi_i(t_i+\tau)-\varphi_i(t_i)}{\tau} - \varphi_i'(t_i)\right|\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n \\
&\qquad\le \eta\int_M \left(\|f(x)\| + \left\|f(x) - f\left(x - \sum_{j=1}^n t_j h_j\right)\right\|\right)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n \\
&\qquad\le \eta\int_M \left(\|f(x)\| + L\sum_{j=1}^n |t_j|\right)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n < \eta\left(\|f(x)\| + \frac{\varepsilon}{2} + L\delta\right)\frac{2\varepsilon}{L2^i},
\end{aligned} \tag{2}$$

where $M = \mathbb{R}^{i-1} \times \left(\left[-\frac{\varepsilon}{2L2^i}, \frac{\varepsilon}{2L2^i}\right] \cup \left[-\frac{\varepsilon}{2L2^i} - \tau, \frac{\varepsilon}{2L2^i} - \tau\right]\right) \times \mathbb{R}^{n-i} \subset \mathbb{R}^n$. We can see that the limit in (1) is uniform with respect to n. Consequently, for any $x \in X$ we have

$$\begin{aligned}
D_{h_i} g(x) &= \lim_{\tau\to 0}\frac{1}{\tau}\big(g(x+\tau h_i) - g(x)\big) = \lim_{\tau\to 0}\lim_{n\to\infty}\frac{1}{\tau}\big(g_n(x+\tau h_i) - g_n(x)\big) \\
&= \lim_{n\to\infty}\lim_{\tau\to 0}\frac{1}{\tau}\big(g_n(x+\tau h_i) - g_n(x)\big) = \lim_{n\to\infty} D_{h_i} g_n(x).
\end{aligned}$$

The limit (1) is uniform with respect to n and so we can interchange the limits above.

Next, the mapping $D_{h_i} g_n$ is L_i-Lipschitz for any $n \ge i$, where $L_i = L\int_{\mathbb{R}} |\varphi_i'(t)|\,\mathrm{d}t$:

$$\begin{aligned}
\|D_{h_i} g_n(x) - D_{h_i} g_n(y)\| &\le \left\|\int_{\mathbb{R}^n}\left(f\left(x - \sum_{j=1}^n t_j h_j\right) - f\left(y - \sum_{j=1}^n t_j h_j\right)\right)\varphi_i'(t_i)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n\right\| \\
&\le L\|x-y\|\int_{\mathbb{R}} |\varphi_i'(t)|\,\mathrm{d}t = L_i\|x-y\|.
\end{aligned}$$

Thus the mapping $D_{h_i} g$ is L_i-Lipschitz for $i = 1, 2, \dots$ This implies that the limit in the definition of the directional derivative $D_{h_i} g(x)$ is uniform with respect to $x \in X$. Indeed,

$$\left\|\frac{1}{\tau}\big(g(x+\tau h_i) - g(x)\big) - D_{h_i} g(x)\right\| = \left\|\frac{1}{\tau}\int_0^\tau D_{h_i} g(x + s h_i)\,\mathrm{d}s - D_{h_i} g(x)\right\| \le \frac{L_i}{|\tau|}\int_0^\tau |s|\,\mathrm{d}s \le L_i|\tau|.$$

Finally, the derivative $D_h g(x)$ exists for any $h \in S_X$ and the limit in the definition is uniform with respect to $x \in X$. To see that, choose $\eta > 0$ and let $i \in \mathbb{N}$ be such that $\|h - h_i\| < \frac{\eta}{L}$. Then for any $\tau \in \mathbb{R} \setminus \{0\}$

$$\left\|\frac{1}{\tau}\big(g(x+\tau h) - g(x)\big) - \frac{1}{\tau}\big(g(x+\tau h_i) - g(x)\big)\right\| \le \frac{L}{|\tau|}\|\tau(h - h_i)\| < \eta.$$

Thus there is $\delta > 0$ such that

$$\left\| \frac{1}{\tau_1}\big(g(x+\tau_1 h) - g(x)\big) - \frac{1}{\tau_2}\big(g(x+\tau_2 h) - g(x)\big) \right\| \le 2\eta + \left\| \frac{1}{\tau_1}\big(g(x+\tau_1 h_i) - g(x)\big) - \frac{1}{\tau_2}\big(g(x+\tau_2 h_i) - g(x)\big) \right\| \le 3\eta$$

for each $x \in X$, $0 < |\tau_1| < \delta$, $0 < |\tau_2| < \delta$. This means that g is UG, provided that for any fixed x the operator $D_h g(x)$ is a bounded linear operator in h.

The fact that $D_{\lambda h} g(x) = \lambda D_h g(x)$ is trivial and the boundedness of the operator follows easily from the Lipschitzness of g. Pick any $i, j \in \mathbb{N}$. Then

$$\begin{aligned}
D_{h_i+h_j} g(x) &= \lim_{\tau\to 0} \frac{1}{\tau}\left(g\big(x+\tau(h_i+h_j)\big) - g(x)\right) = \lim_{\tau\to 0}\lim_{n\to\infty} \frac{1}{\tau}\left(g_n\big(x+\tau(h_i+h_j)\big) - g_n(x)\right) \\
&= \lim_{n\to\infty}\lim_{\tau\to 0} \frac{1}{\tau}\left(g_n\big(x+\tau(h_i+h_j)\big) - g_n(x)\right) \\
&= \lim_{n\to\infty}\lim_{\tau\to 0} \frac{1}{\tau}\left(\int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k + \tau(h_i+h_j)\right)\prod_{k=1}^n \varphi_k(t_k)\,\mathrm{d}\lambda_n - \int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k\right)\prod_{k=1}^n \varphi_k(t_k)\,\mathrm{d}\lambda_n\right) \\
&= \lim_{n\to\infty}\lim_{\tau\to 0} \int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k\right)\frac{\varphi_i(t_i+\tau)\varphi_j(t_j+\tau) - \varphi_i(t_i)\varphi_j(t_j)}{\tau}\prod_{\substack{k=1\\k\ne i,j}}^n \varphi_k(t_k)\,\mathrm{d}\lambda_n \\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k\right)\varphi_i'(t_i)\prod_{\substack{k=1\\k\ne i}}^n \varphi_k(t_k)\,\mathrm{d}\lambda_n + \int_{\mathbb{R}^n} f\left(x - \sum_{k=1}^n t_k h_k\right)\varphi_j'(t_j)\prod_{\substack{k=1\\k\ne j}}^n \varphi_k(t_k)\,\mathrm{d}\lambda_n\right) \\
&= \lim_{n\to\infty}\big(D_{h_i} g_n(x) + D_{h_j} g_n(x)\big) = D_{h_i} g(x) + D_{h_j} g(x).
\end{aligned}$$

Note that we can show, similarly as in (2), that $\lim_{\tau\to 0}$ is uniform with respect to n. Hence we can interchange the limits. Now, for arbitrary $u, v \in X$ and $\eta > 0$, we have

$$\begin{aligned}
\|D_u g(x) - D_v g(x)\| &\le \left\| D_u g(x) - \frac{1}{\tau}\big(g(x+\tau u) - g(x)\big)\right\| \\
&\quad + \left\| \frac{1}{\tau}\big(g(x+\tau u) - g(x+\tau v)\big)\right\| + \left\| D_v g(x) - \frac{1}{\tau}\big(g(x+\tau v) - g(x)\big)\right\| \\
&\le \eta + L\,\|u - v\|
\end{aligned}$$

for τ small enough. Thus $\|D_u g(x) - D_v g(x)\| \le L\,\|u-v\|$. Choose h_i and h_j such that $\|u - h_i\| < \eta$ and $\|v - h_j\| < \eta$. Then

$$\begin{aligned}
\|D_{u+v} g(x) - D_u g(x) - D_v g(x)\| &\le \|D_{u+v} g(x) - D_{h_i+h_j} g(x)\| + \|D_{h_i} g(x) - D_u g(x)\| + \|D_{h_j} g(x) - D_v g(x)\| \\
&\le L\big(\|u+v-h_i-h_j\| + \|u-h_i\| + \|v-h_j\|\big) \le 4L\eta
\end{aligned}$$

for an arbitrary $\eta > 0$.

We have shown that the directional derivatives of g form bounded linear operator and hence g is Gâteaux differentiable. Moreover, since the limits defining the directional derivatives are uniform for $x \in X$, the mapping g is uniformly Gâteaux differentiable. □

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