$\underline{q>1}$: $\lim\limits_{n\to\infty} q^n = +\infty$

$h = q-1>0$, tj. $q = 1+h$

$$q^n = (1+h)^n = 1+nh+\dots+h^n \geq \underbrace{1+n\cdot h}_{\substack{\downarrow\, n\to\infty \\ +\infty}}$$

$\overset{\text{1 policaja}}{\Longrightarrow}$ $\lim q^n = +\infty$

$\underline{q=1}$: $\lim q^n = 1$

$\underline{q\in(0,1)}$: $p=\frac{1}{q}>1$

$$q^n = \frac{1}{\frac{1}{q^n}} = \frac{1}{p^n}, \text{ vemo, že } \lim p^n = +\infty$$

$$\lim q^n \underset{\cancel{?}}{\overset{AL}{=}} \frac{1}{\lim p^n} = \frac{1}{+\infty} = 0$$

$\underline{q=0}$: $\lim q^n = 0$

$q \in (-1,0)$: $\quad |q| \in (0,1)$ $\quad \lim |q^n| = \lim |q|^n = 0$

$|q|^n = |q^n|$ $\quad \Rightarrow \lim q^n = 0$

$q = -1$ $\quad \lim q^n = \lim (-1)^n$ neexistuje

$q < -1$: $\quad \lim q^n$ neexistuje: $\quad \lim q^{2n} = \lim (q^2)^n = +\infty$

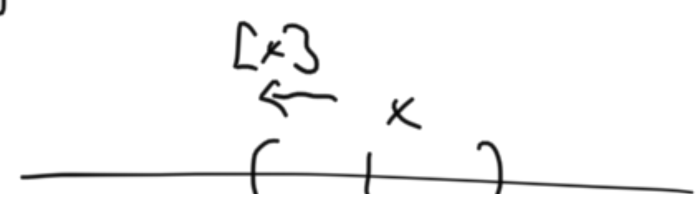
$(q^2 > 1)$

$\lim q^{2n+1} \underset{A2}{=} q \cdot \lim q^{2n} = q \cdot (+\infty) = -\infty$

$\downarrow$

dle věty o limitě vybrané posl.

$$\lim_{n\to\infty} q^n = \begin{cases} +\infty & \ldots\ q > 1 \\ 1 & \ldots\ q = 1 \\ 0 & \ldots\ q \in (-1,1) \\ \text{neexistuje} & \ldots\ q \leq -1 \end{cases}$$



$[\sqrt{\ }$

$$\lim \underbrace{\frac{[\sqrt{n}]}{\sqrt{n}}}_{a_n}$$

$n \qquad n+1$

$$\boxed{x-1 \le [x] \le x}$$

$\forall n \in \mathbb{N}$: $\sqrt{n}-1 \le [\sqrt{n}] \le \sqrt{n}$

$$1-\frac{1}{\sqrt{n}} = \frac{\sqrt{n}-1}{\sqrt{n}} \le a_n \le \frac{\sqrt{n}}{\sqrt{n}} = 1$$

$\sqrt{n} \to +\infty$

$\downarrow$ AL

1

$a_n \xrightarrow{\text{2 POL.}} 1$

$1 \to 1$

$$\lim \sqrt[n]{2^n + \underbrace{5^n}_{\text{dom. člen}}}$$

$\sqrt[n]{a_n}$ | $\sqrt[n]{a}$, a konst. — známá limita

$\sqrt[n]{n}$

$\sqrt[n]{5^n} = 5$

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{2^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{2\cdot 5^n} = 5\cdot \underbrace{\sqrt[n]{2}}_{\to 1}$$

$\downarrow$

5

2 POL.

$$\downarrow \qquad\qquad \downarrow \text{AL}$$
$$5 \qquad\qquad 5$$

jiný způsob:

$$\sqrt[n]{2^n+5^n} = \sqrt[n]{5^n\cdot\left(\left(\tfrac{2}{5}\right)^n+1\right)} = 5\cdot\underbrace{\sqrt[n]{1+\left(\tfrac{2}{5}\right)^n}}_{\sqrt[n]{a_n},\ a_n\ \text{není konstantní}} \xrightarrow{\text{AL}} 5$$

$$1 \le \sqrt[n]{1+\underbrace{\left(\tfrac{2}{5}\right)}_{\le 1}^{n}} \le \sqrt[n]{2}$$

$$\downarrow \qquad \downarrow \text{2POL.} \qquad \downarrow$$
$$1 \qquad 1$$

$\lim \frac{n}{2^n}$

<u>Pomocné tvrzení</u>:

$\{a_n\}$ posl. kladných čísel, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

Pak $\lim a_n = 0$.

↳ dle věty o lim. a nerovnostech:

$\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \quad \frac{a_{n+1}}{a_n} < 1 \quad / \cdot a_n$

$a_{n+1} < a_n$

Tedy $\{a_n\}$ je od n_0 klesající.

Dle věty o lim. monot. posl. existuje $\lim a_n = A$.

$a_n \geq 0 \Rightarrow \lim a_n \neq -\infty$ (dokonce $\lim a_n \geq 0$)

$\lim a_n \neq +\infty$

$\{a_{n+1}\}$ vybraná z $\{a_n\}$

AL $\Rightarrow$ $\lim \frac{a_{n+1}}{a_n} \underset{A \neq 0}{\overset{?}{=}} \frac{\lim a_{n+1}}{\lim a_n} = \frac{A}{A} = 1$

spor

Tedy $A = 0$.

$\lim \frac{n}{2^n} = 0$

Pom. kritérium: $\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} =$

$= \lim \frac{n+1}{n} \cdot \frac{1}{2} \overset{AL}{=} \frac{1}{2} < 1$

$$\lim \frac{10^n}{n!} = 0 \qquad \lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \lim \frac{10}{n} = 0 \;\checkmark$$

$$\boxed{\begin{array}{l} k \in \mathbb{N},\ q > 1 \\ \lim \frac{n^k}{q^n} = 0, \quad \lim \frac{q^n}{n!} = 0 \end{array}}$$

růstová škála: „$n^k \ll q^n \ll n!$" $\left(\ll n^n\right)$

$$\lim \frac{n^5 + 2^n + 17^n}{n! + n + 3^n} = \lim \frac{\cancel{n!}\left(\underbrace{\frac{n^5}{n!}}_{\to 0} + \underbrace{\frac{2^n}{n!}}_{\to 0} + \underbrace{\frac{17^n}{n!}}_{\to 0}\right)}{\cancel{n!}\left(1 + \underbrace{\frac{n}{n!}}_{\to 0} + \underbrace{\frac{3^n}{n!}}_{\to 0}\right)} \overset{AL}{=} \frac{0}{1} = 0$$

růstová škála

$$\lim \underbrace{\frac{\left(2 + \frac{1}{n}\right)^{100} - \left(4 - \frac{3}{n}\right)^{50}}{\left(8 - \frac{1}{n}\right)^{34} - \left(4 + \frac{1}{n}\right)^{51}}}_{a_n} \overset{AL}{=} \cancel{\frac{2^{100} - 4^{50}}{8^{34} - 4^{51}} = \frac{2^{100} - 2^{100}}{2^{102} - 2^{102}}} = \frac{0}{0}$$

8 $\qquad\qquad\qquad$ 4

$$\left(2+\frac{1}{n}\right)^{100} = 2^{100} + \underline{100 \cdot 2^{99} \cdot \frac{1}{n}} + a_2 \cdot \frac{1}{n^2} + a_3 \cdot \frac{1}{n^3} + \dots + a_{100} \cdot \frac{1}{n^{100}}$$

$$\left(4-\frac{3}{n}\right)^{50} = 4^{50} - \underbrace{50 \cdot 4^{49} \cdot \frac{3}{n}} + b_2 \cdot \frac{1}{n^2} + b_3 \cdot \frac{1}{n^3} + \dots + b_{50} \cdot \frac{1}{n^{50}}$$

$$100 \cdot 2 \cdot 4^{48} \cdot \frac{3}{n}$$

$$\parallel$$

$$100 \cdot 2^{2 \cdot 48 + 1} \cdot \frac{3}{n}$$

$$\parallel$$

$$\underline{100 \cdot 2^{97} \cdot \frac{3}{n}}$$

$c_2, \dots, c_{100} \in \mathbb{R}$

$$\left(2+\frac{1}{n}\right)^{100} - \left(4-\frac{3}{n}\right)^{50} = 100 \cdot 2^{97} \cdot (\underbrace{4+3}_{7}) \cdot \frac{1}{n} + c_2 \cdot \frac{1}{n^2} + c_3 \frac{1}{n^3} + \dots + c_{100} \cdot \frac{1}{n^{100}}$$

$$\left(8-\frac{1}{n}\right)^{34} = \cancel{8^{34}} - 34 \cdot \underset{\substack{\parallel \\ 2^{99}}}{8^{33}} \cdot \frac{1}{n} + \underbrace{\alpha_2 \cdot \frac{1}{n^2} + \dots + \alpha_{34} \cdot \frac{1}{n^{34}}}$$

$\cancel{\alpha_2 \frac{1}{n^2} + \dots}$

$$\left(4+\frac{1}{n}\right)^{51} = \cancel{4^{51}} + 51 \cdot \underset{\substack{\parallel \\ 2^{100}}}{4^{50}} \cdot \frac{1}{n} + \beta_2 \frac{1}{n^2} + \dots + \beta_{51} \cdot \frac{1}{n^{51}}$$

$$\left(8-\frac{1}{n}\right)^{34} - \left(4+\frac{1}{n}\right)^{51} = -2^{99} (34 + 2 \cdot 51) \cdot \frac{1}{n} + d \frac{1}{\dots} + \dots + d_{\dots} \cdot \frac{1}{\dots}$$

$$\lim a_n = \lim \frac{\cancel{\frac{1}{n}}\left(700\cdot 2^{97} + \overbrace{c_2\cdot\frac{1}{n}}^{\to 0} + \dots + \overbrace{c_{100}\cdot\frac{1}{n^{99}}}^{\to 0}\right)}{\cancel{\frac{1}{n}}\left(-136\cdot 2^{99} + \underbrace{d_2\cdot\frac{1}{n}}_{\downarrow\, 0} + \dots + \underbrace{d_{51}\cdot\frac{1}{n^{50}}}_{\downarrow\, 0}\right)} \overset{AL}{=} \frac{700\cdot 2^{97}}{-136\cdot 2^{99}} =$$

$$= -\frac{700}{4\cdot 136} = -\frac{175}{136}$$

$$\lim \frac{\overbrace{\left[\sqrt[3]{n^3+1}\right]}^{\to+\infty} + \overbrace{\left[\sqrt[3]{n^3-1}\right]}^{\to+\infty}}{\underbrace{\sqrt[n]{\underbrace{1+2^n+\dots\dots+n^n}}}_{\to+\infty}} \qquad a_n$$

n sčítanců, tedy počet sčítanců se mění s rostoucím n!

$$\geq \sqrt[n]{n^n} = n$$

$$n^n \leq \underset{n^n}{\overset{\vee\!\mid}{1}} + \underset{n^n}{\overset{\vee\!\mid}{2^n}} + \dots + \underset{n^n}{\overset{\vee\!\mid}{n^n}} \leq n^n + n^n + \dots + n^n = n\cdot n^n$$

$$n \le \sqrt{\ldots\ldots\ldots} \le n\cdot\sqrt[n]{n}$$

$$\sqrt[3]{n^3+1}-1+\sqrt[3]{n^3-1}-1 \le \left[\sqrt[3]{n^3+1}\right]+\left[\sqrt[3]{n^3-1}\right] \le \sqrt[3]{n^3+1}+\sqrt[3]{n^3-1}$$

$$n\left(\sqrt[3]{1+\frac{1}{n^3}}+\sqrt[3]{1-\frac{1}{n^3}}-\frac{2}{n}\right) \qquad\qquad n\left(\sqrt[3]{1+\frac{1}{n^3}}+\sqrt[3]{1-\frac{1}{n^3}}\right)$$

$$\frac{\cancel{n}\left(\overbrace{\sqrt[3]{1+\frac{1}{n^3}}+\sqrt[3]{1-\frac{1}{n^3}}}^{\to 2}-\overbrace{\frac{2}{n}}^{\to 0}\right)}{\cancel{n}\cdot\underbrace{\sqrt[n]{n}}_{\to 1}} \le a_n \le \frac{\cancel{n}\left(\overbrace{\sqrt[3]{1+\frac{1}{n^3}}}^{\to 1\ \text{AL}}+\overbrace{\sqrt[3]{1-\frac{1}{n^3}}}^{\to 1}\right)}{\cancel{n}}$$

$1 \leftarrow$

$\sqrt[3]{a_n}\to\sqrt[3]{a}$

$\to 1$ AL

$$\begin{array}{ccc} \downarrow \text{AL} & \downarrow \text{2 POL.} & \downarrow \text{AL} \\ 2 & 2 & 2 \end{array}$$