$$\lim_{n\to\infty} \frac{1}{n^3} = 0$$

1) z def. nechť $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$

z arch. vl. $\exists n_0 \in \mathbb{N}$: $n_0 > \frac{1}{\sqrt[3]{\varepsilon}}$

$\forall n \in \mathbb{N}, n \geq n_0$:

$$\left| \frac{1}{n^3} - 0 \right| = \frac{1}{n^3} \leq \frac{1}{n_0^3} < \varepsilon .$$

2) $\forall n \in \mathbb{N}$: $\frac{1}{n^3} \leq \frac{1}{n}$

$n^3 \geq n$

$n^2 \geq 1$ ✓

$$\forall n \in \mathbb{N}: \quad \underset{a_n}{0} \leq \underset{c_n}{\frac{1}{n^3}} \leq \underset{b_n}{\frac{1}{n}}$$

$$\left.\begin{array}{l} \lim a_n = 0 \\ \lim b_n = 0 \end{array}\right\} \overset{\text{2POL.}}{\Rightarrow} \lim c_n = 0$$

$$\forall n \in \mathbb{N}: \quad 0 \leq \frac{1}{n^3} \leq \frac{1}{n}$$

$$0 \to 0, \quad \frac{1}{n} \to 0 \quad \overset{\text{2POL}}{\Rightarrow} \quad \frac{1}{n^3} \to 0$$

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3) $$\frac{1}{n^3} = \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n}$$

$$\lim \frac{1}{n^3} = \lim \left(\frac{1}{n}\cdot\frac{1}{n}\cdot\frac{1}{n}\right) \underset{\cancel{?}}{\overset{AL}{=}} \left(\lim \frac{1}{n}\right)\cdot\left(\lim \frac{1}{n}\right)\cdot\left(\lim \frac{1}{n}\right) = 0\cdot 0\cdot 0 = 0$$

$\Uparrow$ je def.

$$\left(\lim \left(\frac{1}{n}\cdot\left(\frac{1}{n}\cdot\frac{1}{n}\right)\right) = \left(\lim \frac{1}{n}\right)\cdot \underline{\lim \left(\frac{1}{n}\cdot\frac{1}{n}\right)}\right)$$

$$\lim_{n\to\infty} \frac{2+\frac{1}{n}-\frac{7}{n^5}}{7-\frac{3}{n^2}} \underset{\cancel{?}}{\overset{AL}{=}} \frac{\lim\left(2+\frac{1}{n}-\frac{7}{n^5}\right)}{\lim\left(7-\frac{3}{n^2}\right)} \underset{\cancel{?}}{\overset{AL}{=}} \frac{\lim 2 + \lim \frac{1}{n} + \lim\left(-\frac{7}{n^5}\right)}{\lim 7 + \lim\left(-\frac{3}{n^2}\right)} =$$

$$\underset{\cancel{?}}{\overset{AL}{=}} \frac{\lim 2 + \lim \frac{1}{n} + \lim(-7)\cdot \lim \frac{1}{n}\cdot \lim \frac{1}{n}\cdot \lim \frac{1}{n}\cdot \lim \frac{1}{n}\cdot \lim \frac{1}{n}}{\lim 7 + \lim(-3)\cdot\left(\lim \frac{1}{n}\right)^2} =$$

$$= \frac{\overbrace{2+0+\boxed{(-7)\cdot 0^5}}}{\underbrace{7+\boxed{(-3)\cdot 0^2}}} = \boxed{\frac{2}{7}}$$

$$\lim \frac{2+\overset{\nearrow 0}{\frac{1}{n}}-\overset{\nearrow 0}{\frac{7}{n^5}}}{7-\frac{3}{n^2}} \overset{AL}{=} \frac{2}{7}$$

$\to 0$

není def.!

$$\lim \frac{n+7}{5n+3} \cancel{\overset{AL}{=}}_{?} \frac{\lim (n+7)}{\lim (5n+3)} \overset{AL}{\underset{\cancel{?}}{=}} \frac{\lim n + \lim 7}{(\lim 5)\cdot(\lim n) + \lim 3} = \frac{+\infty + 7}{\underbrace{5\cdot(+\infty)+3}_{+\infty}} = \boxed{\frac{+\infty}{+\infty}}$$

pro obrovská n „skoro stejné" jako n

$$\lim \frac{n+7}{5n+3} = \lim \frac{\cancel{n}(1+\frac{7}{n})}{5\cancel{n}(1+\frac{3}{5n})} = \lim \frac{1+\frac{7}{n}}{5(1+\frac{3}{5n})} \overset{AL}{\underset{\cancel{?}}{=}} \frac{1}{5(1+0)} = \frac{1}{5}$$

pro obrovská n je to „skoro stejné" jako $5n$

$$\lim \frac{\cancel{n}}{5\cancel{n}} = \lim \frac{1}{5} = \frac{1}{5}$$

dominantní člen

dom. n dom. $2n$

$$\lim \frac{(n+1)(2n+7)}{n^2+5n+15} \overset{AL}{\underset{?}{=}} \cancel{\frac{\lim(n+1)\cdot\lim(2n+7)}{\lim n^2 + \lim 5n + 15} = \frac{(+\infty)\cdot(+\infty)}{(+\infty)+(+\infty)+15} = \frac{+\infty}{+\infty}}$$

dom. n^2

$$\lim \frac{n\cdot 2n}{n^2} = \lim \frac{2}{1} = 2$$

$$\lim \frac{\cancel{n}(1+\frac{1}{n})\cdot \cancel{n}\cdot(2+\frac{7}{n})}{\ldots} = \ldots \frac{(1+\frac{1}{n})\cdot(2+\frac{7}{n})}{\ldots} \overset{AL}{=} (1+0)(2+0)\ldots$$

$$\frac{\ldots}{\cancel{n^2}\left(1+\frac{5}{n}+\frac{15}{n^2}\right)} = \lim \frac{\ldots}{1+\underbrace{\frac{5}{n}}_{\downarrow \atop 0}+\underbrace{\frac{15}{n^2}}_{\downarrow \text{ AL} \atop 0}} \overset{AL}{=} \frac{\ldots}{1+0+0} = 2$$

$$\lim \sqrt{2+\frac{1}{n}} \overset{\cancel{AL}}{=} \sqrt{2}$$

$$\lim 2+\frac{1}{n} \overset{AL}{=} 2+0 = 2$$

CHCI: $\lim \left|\sqrt{2+\frac{1}{n}}-\sqrt{2}\right| = 0$

$$0 \le \left|\sqrt{2+\frac{1}{n}}-\sqrt{2}\right| = \left(\sqrt{2+\frac{1}{n}}-\sqrt{2}\right)\cdot\frac{\sqrt{2+\frac{1}{n}}+\sqrt{2}}{\sqrt{2+\frac{1}{n}}+\sqrt{2}} = \frac{\left(2+\frac{1}{n}\right)-2}{\sqrt{2+\frac{1}{n}}+\sqrt{2}} =$$

$0 <_{\forall n\in\mathbb{N}}$

$$(a-b)\cdot\frac{a+b}{a+b}$$

$$a^2-b^2 = (a-b)(a+b)$$

$$= \frac{\frac{1}{n}}{\underbrace{\sqrt{2+\frac{1}{n}}+\sqrt{2}}_{\ge 1}} \le \frac{1}{n}$$

$$\lim \frac{\frac{1}{n}}{\sqrt{2+\frac{1}{n}}+\sqrt{2}} \overset{AL}{=} \frac{\overbrace{\lim \frac{1}{n}}^{\to 0}}{\underbrace{\lim\sqrt{2+\frac{1}{n}}+\lim\sqrt{2}}_{2}}$$

co když …

$\forall n\in\mathbb{N}: \quad 0 \le \left|\sqrt{2+1\ldots}-\sqrt{2}\right| \le 1\ldots$

$0 \le \left|\sqrt{2+\frac{1}{n}}-\sqrt{2}\right| \le \frac{1}{n}$

$\downarrow \quad\quad \downarrow_{\text{2POL}} \quad\quad \downarrow$

$0 \quad\quad 0 \quad\quad 0$

neexistuje?

TVRZENÍ: $\{a_n\}$, $a_n \geq 0$, $\lim a_n = A \in \mathbb{R}^*$,

$A \neq -\infty$ (věta o lim a uspořádání) $A \geq 0$

Pak $\lim \sqrt{a_n} = \begin{cases} \sqrt{A}, & A \in \mathbb{R} \\ +\infty, & A = +\infty \end{cases}$.

Důkaz: nechť $A \neq 0$, $A \neq +\infty$.

$$0 \le \left|\sqrt{a_n}-\sqrt{A}\right| = \left|\sqrt{a_n}-\sqrt{A}\right| \cdot \frac{\sqrt{a_n}+\sqrt{A}}{\sqrt{a_n}+\sqrt{A}} = \frac{|a_n-A|}{\underbrace{\sqrt{a_n}+\sqrt{A}}_{\geq \sqrt{A}}} \le \frac{|a_n-A|}{\sqrt{A}}$$

$\sqrt{a_n}+\sqrt{A} \neq 0$

$$\lim \frac{|a_n-A|}{\sqrt{A}} \overset{AL}{=} \frac{1}{\sqrt{A}} \cdot \lim |a_n - A| = 0$$

$$\forall n \in \mathbb{N}: \quad 0 \le \left|\sqrt{a_n}-\sqrt{A}\right| \le \frac{|a_n-A|}{\sqrt{A}}$$

$\downarrow \quad\quad\quad\quad \downarrow$

$0 \qquad \Big\downarrow \text{ zpol.} \qquad 0$

0

$\underline{A=0}$ nechť $\varepsilon > 0$. $\lim a_n = 0$:

$\exists n_0 \in \mathbb{N}: \forall n \geq n_0: |a_n - 0| < \varepsilon^2$, kde $|a_n - 0| = a_n$

CHCI $|\sqrt{a_n} - 0| < \varepsilon$, kde $|\sqrt{a_n} - 0| = \sqrt{a_n}$

$a_n < \varepsilon^2$

$\Rightarrow \sqrt{a_n} < \varepsilon$

$\underline{A=+\infty}$ nechť $K \in \mathbb{R}$.

CHCI $\sqrt{a_n} > K$ $\left\langle \begin{array}{l} K < 0, \text{ pak platí } \forall n \in \mathbb{N} \\ K \geq 0 \end{array} \right.$

$\lim a_n = +\infty$: $\exists n_0 \in \mathbb{N}\ \forall n \geq n_0: a_n > K^2 \Rightarrow \sqrt{a_n} > K$

Pozn.: analogické tvrzení platí i pro jiné odmocniny:

Je-li $k \in \mathbb{N}$, $k \geq 2$ a $\{a_n\}$, $a_n \geq 0$, $\lim a_n = A \in \mathbb{R}^*$.

Pak $\lim \sqrt[k]{a_n} = \left\langle \begin{array}{ll} \sqrt[k]{A} & \ldots A \in \mathbb{R} \\ +\infty & \ldots A = +\infty \end{array} \right.$