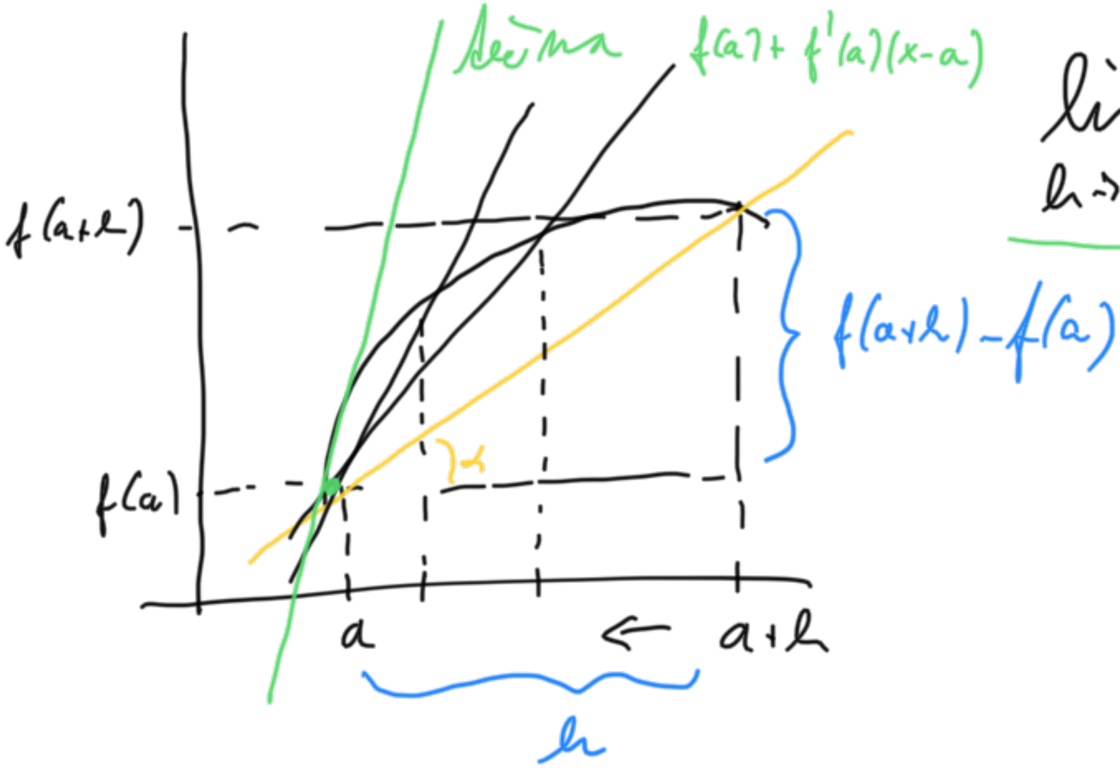


$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f'(a)$ ... směrnice tečny ke grafu f v bodě $[a, f(a)]$

$\operatorname{tg}\alpha = \dfrac{f(a+h)-f(a)}{h}$ ... směrnice sečny

Poznámky:

1) $f'(a) = $
- neexistuje
- existuje
  - vlastní, tj. $\in \mathbb{R}$
  - nevlastní, tj. $+\infty$, $-\infty$

2) $f'(a)$ existuje $\Rightarrow$ f je definována na nějakém okolí bodu a.

3) $f'(a)$ existuje $\Leftrightarrow$ $f'_+(a)$ a $f'_-(a)$ existují a rovnají se.

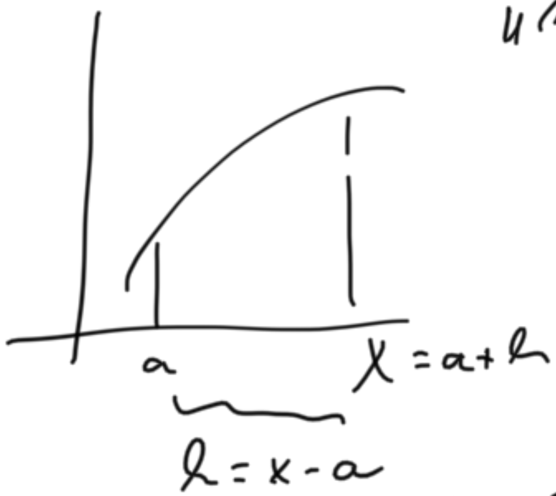
Pak $f'(a) = f'_+(a) = f'_-(a)$.

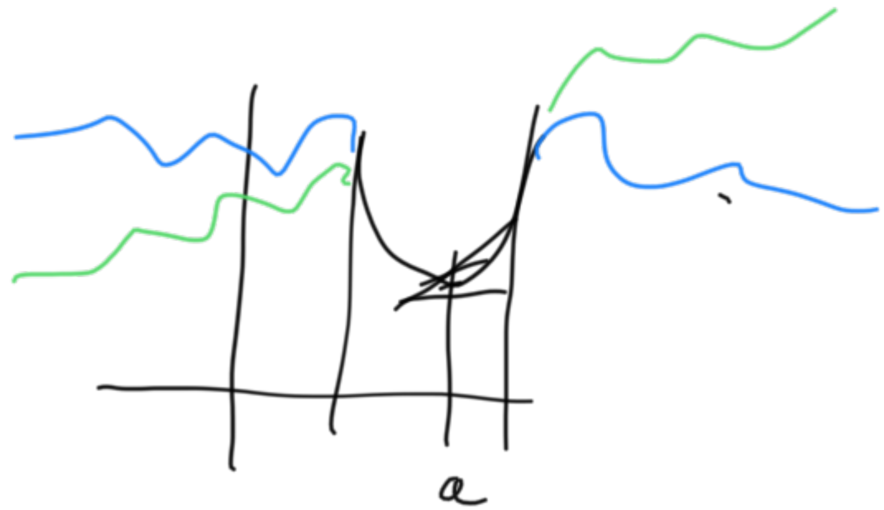
4)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \underset{\text{VoLSF (P)}}{=} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

„substituce" $h = x - a$

afinní





5) Derivace je „lokální pojem".

Př.:

1) $f(x) = c, \quad x \in B(a, \delta), \quad a \in \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \boxed{\frac{\overset{0}{c - c}}{h}} = \lim_{h \to 0} 0 = 0$$

2) $f(x) = x^n, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \quad a \in \mathbb{R}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{\cancel{(x-a)} \overbrace{\left(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}\right)}^{n \text{ sčítanců}}}{\cancel{x - a}} =$$

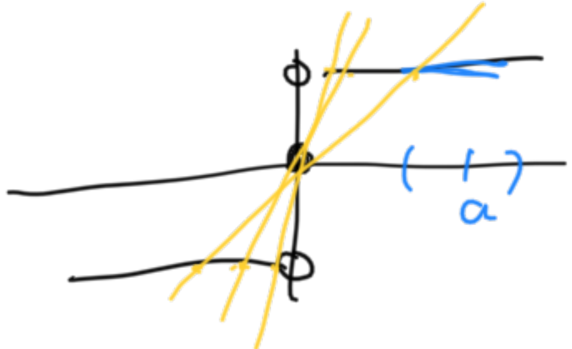
$$a^{n-1} + a^{n-1} + \dots + a^{n-1} = n a^{n-1}$$

$$\underset{AL}{=} a^{n} + a^{\cdots} + \dots + a^{\cdots} = n \cdot a$$

(spojitost polynomu)

3) $f(x) = \operatorname{sgn} x$

dle 1) je $f'(a) = 0$ pro $a \neq 0$

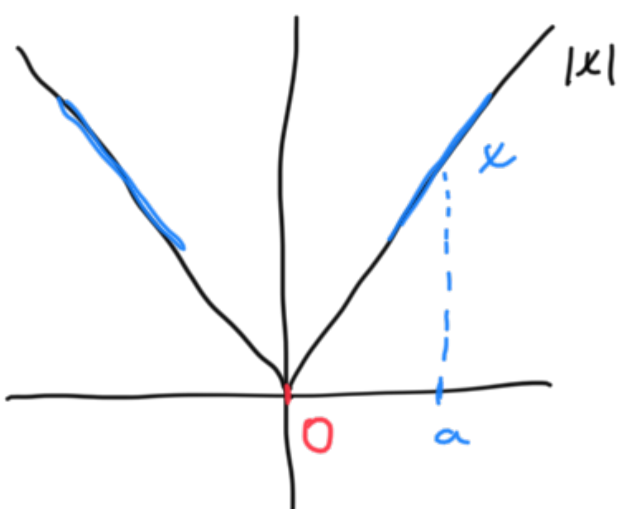


$$f'_{+}(0) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{\operatorname{sgn} h - 0}{h} =$$

$$= \lim_{h \to 0\pm} \frac{\operatorname{sgn} h}{h} = \begin{cases} \lim\limits_{h \to 0+} \dfrac{1}{h} = +\infty \\ \lim\limits_{h \to 0-} \dfrac{-1}{h} = +\infty \end{cases}$$

$$\Rightarrow f'(0) = +\infty$$

4) $f(x) = |x|$



$a \in (0, +\infty) \dots f'(a) = 1$

$(f(x) = x$ pro $x \in (0, +\infty))$

$a \in (-\infty, 0) \dots f'(a) = -1$

$(f(x) = -x$ pro $x \in (-\infty, 0))$

$$\lim \frac{|h|}{h} = \lim \frac{h}{h} =$$

$$\dots |0+h| - |0| \dots$$

$$f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} \cancel{=} \begin{cases} \lim\limits_{h\to 0+} \ldots = 1 \\ \lim\limits_{h\to 0-} \frac{|h|}{h} = \lim\limits_{h\to 0-} \frac{-h}{h} = -1 \end{cases}$$

$$\left.\begin{aligned} f'_+(0) &= 1 \\ f'_-(0) &= -1 \end{aligned}\right\} \Rightarrow f'(0) \text{ neexistuje}$$

<u>Důkaz</u>:

$$\lim_{x\to a} f(x)-f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a) \overset{AL}{\underset{\cancel{?}}{=}} \overbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}^{f'(a)} \cdot \underbrace{\lim_{x\to a}(x-a)}_{0} =$$

$$= \underbrace{f'(a)}_{\in \mathbb{R}} \cdot 0 = 0$$

<u>Poznámky</u>: 1) Pokud $f'(a) = +\infty$ $(-\infty)$, pak f nemusí být v a spojitá.

(např. fce sgn)

2) Věta platí i v jednostranné verzi.

3) Věta opačná neplatí. Viz $f(x)=|x|$.

<u>Důkaz:</u> (i)

$$(f+g)'(a) = \lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h} = \lim_{h\to 0} \frac{f(a+h)+g(a+h)-\left(f(a)+g(a)\right)}{h} =$$

$$= \lim_{h\to 0} \frac{\left(f(a+h)-f(a)\right) + \left(g(a+h)-g(a)\right)}{h} \overset{AL}{=} f'(a)+g'(a)$$

$f'(a)$ $\qquad$ $g'(a)$

(iii) $$(f\cdot g)'(a) = \lim_{x\to a} \frac{(f\cdot g)(x)-(f\cdot g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)\cdot g(x)-f(a)\cdot g(a)}{x-a} =$$

$$= \lim_{x\to a} \frac{f(x)g(x)-f(a)\cdot g(x)+f(a)\cdot g(x)-f(a)\cdot g(a)}{x-a} =$$

$$= \lim_{x\to a} \left(\underbrace{g(x)}_{\to\, \mathbb{R} \ni g(a)} \cdot \underbrace{\frac{f(x)-f(a)}{x-a}}_{\to\, f'(a)} + f(a)\cdot \underbrace{\frac{g(x)-g(a)}{x-a}}_{\to\, g'(a)} \right) \overset{AL}{=} g(a)\cdot f'(a) + f(a)\cdot g'(a)$$

$g'(a)$ je vlastní $\underset{V36}{\Rightarrow}$ g je spojitá v a

(iv)

$$\left(\frac{f}{g}\right)'(a) = \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} = \lim_{x\to a} \boxed{\frac{1}{g(x)\cdot g(a)}} \cdot \frac{f(x)\cdot g(a) - f(a)\cdot g(x)}{x-a} =$$

je df. na $B(a,\Delta)$

$\xrightarrow[\text{spoj. } g]{AL} \frac{1}{g(a)\cdot g(a)}$

$$\left.\begin{matrix} g'(a) \text{ je vlastní} \overset{V36}{\Rightarrow} g \text{ je spojitá v } a \\ g(a) \neq 0 \end{matrix}\right\} \Rightarrow \exists \Delta > 0 : \forall x \in B(a,\Delta) : g(x) \neq 0$$

$$= \lim_{x\to a} \frac{1}{g(x)\cdot g(a)} \cdot \frac{f(x)\cdot g(a) - f(a)\cdot g(a) + f(a)\, g(a) - f(a)\cdot g(x)}{x-a} =$$

$$= \lim \frac{1}{\ldots} \cdot \left(\frac{g(a)\cdot (f(x)-f(a))}{\ldots} - \frac{f(a)(g(x)-g(a))}{\ldots} \right) \overset{AL}{=}$$

$$\lim_{x\to a} \frac{1}{g(x)\cdot g(a)} \left(\underbrace{\frac{\cdots}{x-a}}_{\to f'(a)} \;\; \underbrace{\frac{\cdots}{x-a}}_{\to g'(a)} \right)$$

$$= \frac{1}{g(a)^2} \cdot \left(g(a)\cdot f'(a) - f(a)\cdot g'(a) \right) \qquad \square$$

Pozn.: (i) platí obecněji: stačí, aby výraz $f'(a) + g'(a)$ byl definován.

Důkaz:

$$(f\circ g)'(x_0) = \lim_{x\to x_0} \frac{f(g(x)) - f(\overbrace{g(x_0)}^{y_0})}{x - x_0} = \lim_{x\to y_0} \boxed{\frac{f(g(x)) - f(\overbrace{g(x_0)}^{y_0})}{\underbrace{g(x) - g(x_0)}_{y_0}}} \cdot \frac{\overbrace{g(x)-g(x_0)}^{\to g'(x_0)}}{x - x_0} =$$

$$= \lim_{x\to x_0} \boxed{F(g(x))} \cdot \frac{g(x)-g(x_0)}{x-x_0} \overset{AL}{=} f'(y_0)\cdot g'(x_0)$$

vnější fce $\to$ $F(y) = \dfrac{f(y) - f(y_0)}{y - y_0}$ není def. v y_0

„subst." $y = g(x)$ nejde! (S) ?

VOLSF $\lim_{y\to y_0} F(y) = f'(y_0)$

NEMUSÍ BÝT SPLNĚNA (P) !

ŠPATNĚ: nemůžu rozšířit výrazem $g(x)-g(x_0)$, protože může být 0!

---

nápad: potřebuji zařídit, aby byla splněna (S)

Položím $F(y) = \begin{cases} \dfrac{f(y)-f(y_0)}{y-y_0}, & y \in D_f \setminus \{y_0\} \\ \underbrace{f'(y_0)}_{\in\mathbb{R}}, & y = y_0 \end{cases}$

CHCI: F spojitá v y_0, tj. $F(y_0) = \lim\limits_{y\to y_0} F(y) = \lim\limits_{y\to y_0} \dfrac{f(y)-f(y_0)}{y-y_0} = f'(y_0)$

Pak F je spojitá v y_0.

g je spojitá v x_0 (V36)

$\lim\limits_{x\to x_0} g(x) = g(x_0) = y_0$

$\lim\limits_{y\to y_0} F(y) = f'(y_0)$

$$(f\circ g)'(x_0) = \lim_{x\to x_0} \frac{f(g(x))-f(g(x_0))}{x-x_0} = \lim_{x\to x_0} F(g(x))\cdot\frac{g(x)-g(x_0)}{x-x_0} \overset{AL}{\underset{\substack{VOLSF\\(S)}}{=}} \boxed{f'(y_0)\cdot g'(x_0)}$$

TVRDÍM: $\dfrac{f(g(x))-f(g(x_0))}{\ldots} = F(g(x))\cdot\dfrac{g(x)-g(x_0)}{\ldots}$ pro $x \in D_{f\circ g}\setminus\{x_0\}$

$$\overline{\qquad} \quad x - x_0 \qquad\qquad x - x_0$$

$\hookrightarrow$ Je-li $x \in \mathcal{D}_{f \circ g}$ takové, že $\overset{g(x) \neq y_0}{g(x) \neq g(x_0)}$, pak

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = \underbrace{\frac{f(g(x)) - f(\overbrace{g(x_0)}^{y_0})}{g(x) - \underbrace{g(x_0)}_{y_0}}}_{F(g(x))} \cdot \frac{g(x) - g(x_0)}{x - x_0}.$$

VOLSF: vnější fce F vnitřní g „$y = g(x)$“

Je-li $x \neq x_0$, $g(x) = g(x_0)$, pak

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = \frac{f(g(x_0)) - f(g(x_0))}{x - x_0} = 0$$

$$\| $$

$$F(\underbrace{g(x)}_{\substack{\shortparallel \\ g(x_0) \\ \shortparallel \\ y_0}}) \cdot \frac{g(x) - g(x_0)}{x - x_0} = \underbrace{F(y_0)}_{\substack{\shortparallel \\ f'(y_0)}} \cdot \frac{0}{x - x_0} = 0$$

$\square$