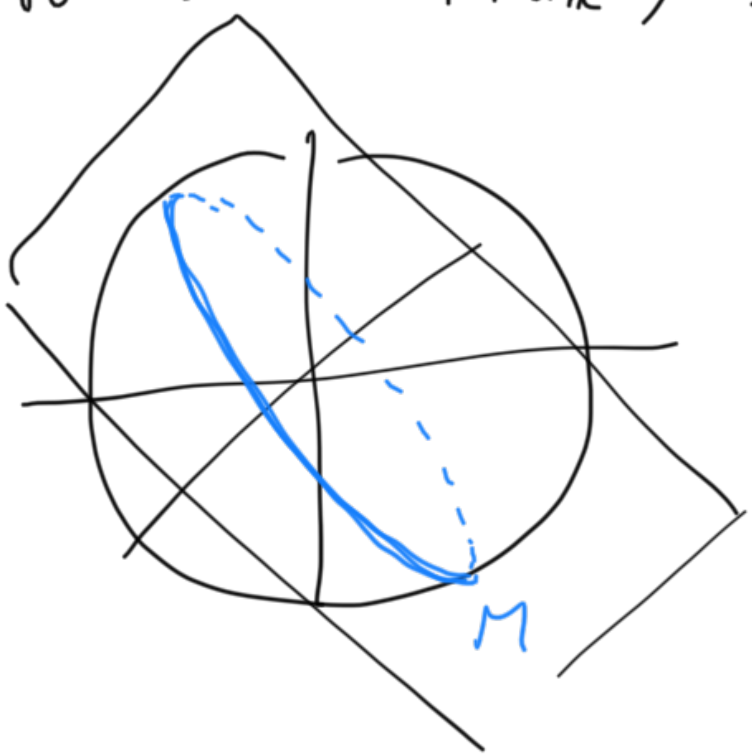
$f(x,y,z) = xyz$, $M = \{[x,y,z] \in \mathbb{R}^3;\ \underbrace{x^2+y^2+z^2=1}_{\text{sféra o poloměru 1}},\ x+y+z=0\}$

$\Rightarrow M$ je omezená

M je uzav. (průnik 2 uzav. množin $\{x^2+y^2+z^2=1\} \cap \{x+y+z=0\}$

M je kompaktní

f je spojitá na M (na $\mathbb{R}^3$) $\Big\} \Rightarrow f$ nabývá na M extrémů



$$M = \{[x,y,z] \in \overset{G}{\overset{\shortparallel}{\mathbb{R}^3}};\ g_1(x,y,z)=0,\ g_2(x,y,z)=0\}$$

$g_1(x,y,z) = x^2+y^2+z^2-1$

$g_2(x,y,z) = x+y+z$

$f \in C^1(G)$

$g_1, g_2 \in C^1(G)$

$\nabla f(x,y,z) = [yz, xz, xy]$

$\nabla g_1(x,y,z) = [2x, 2y, 2z]$

$\nabla g_2(x,y,z) = [1, 1, 1]$

Lagr. věta o multiplikátorech:

Buď (I) $\nabla g_1, \nabla g_2$ LZ

nebo (II) $\exists \lambda_1, \lambda_2 \in \mathbb{R}$:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$$

(I) Vektory $[2x, 2y, 2z]$ a $[1,1,1]$ (≠0) jsou LZ ⟺

$$\exists c \in \mathbb{R}: \ [2x, 2y, 2z] = c \cdot [1,1,1] = [c,c,c]$$

$$\iff x = y = z$$

ale bod $[x,x,x] \notin M$ pro žádné $x \in \mathbb{R}$: $x+x+x=0 \Rightarrow x=0$

$0^2+0^2+0^2 \neq 1$

(II) do souřadnic:

$$yz + \lambda_1 2x + \lambda_2 \cdot 1 = 0$$
$$xz + \lambda_1 \cdot 2y + \lambda_2 = 0$$
$$xy + \lambda_1 \cdot 2z + \lambda_2 = 0$$

$$\boxed{\begin{aligned} x^2+y^2+z^2 &= 1 \\ x+y+z &= 0 \end{aligned}}$$

odečtu 2. od 1.:

$$yz - xz + \lambda_1(2x-2y) = 0$$

$$z(y-x) + 2\lambda_1(x-y) = 0$$

$$\underline{(x-y)(2\lambda_1 - z) = 0} \quad \begin{cases} x = y \\ z = 2\lambda_1 \end{cases}$$

odečtu 3. od 2.:

$$x(z-y) + 2\lambda_1(y-z) = 0$$

$$(y-z)(2\lambda_1 - x) = 0 \quad \begin{cases} y = z \\ x = 2\lambda_1 \end{cases}$$

$$\boxed{\begin{array}{c} x = y \\ \text{nebo} \\ y = z \\ \text{nebo} \\ x = z \end{array}}$$

a) $x = y$:

$$2y^2 + z^2 = 1$$
$$\underline{2y + z = 0}$$

$$z = -2y$$
$$2y^2 + 4y^2 = 1$$
$$y^2 = \frac{1}{6}$$
$$\underline{y = \pm\frac{1}{\sqrt{6}} = x}, \quad z = \mp\frac{2}{\sqrt{6}}$$

Body podezřelé z extrému:

$$\boxed{\left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right]}, \quad \boxed{\left[-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right]}$$

$f = -\frac{2}{6\sqrt{6}} = -\frac{1}{3\sqrt{6}}$ $\qquad$ $f = \frac{1}{3\sqrt{6}}$

b) $y=z$

P.B. $\left[-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$, $\left[\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]$

$f = -\frac{1}{3\sqrt{6}}$ $\qquad$ $f = \frac{1}{3\sqrt{6}}$

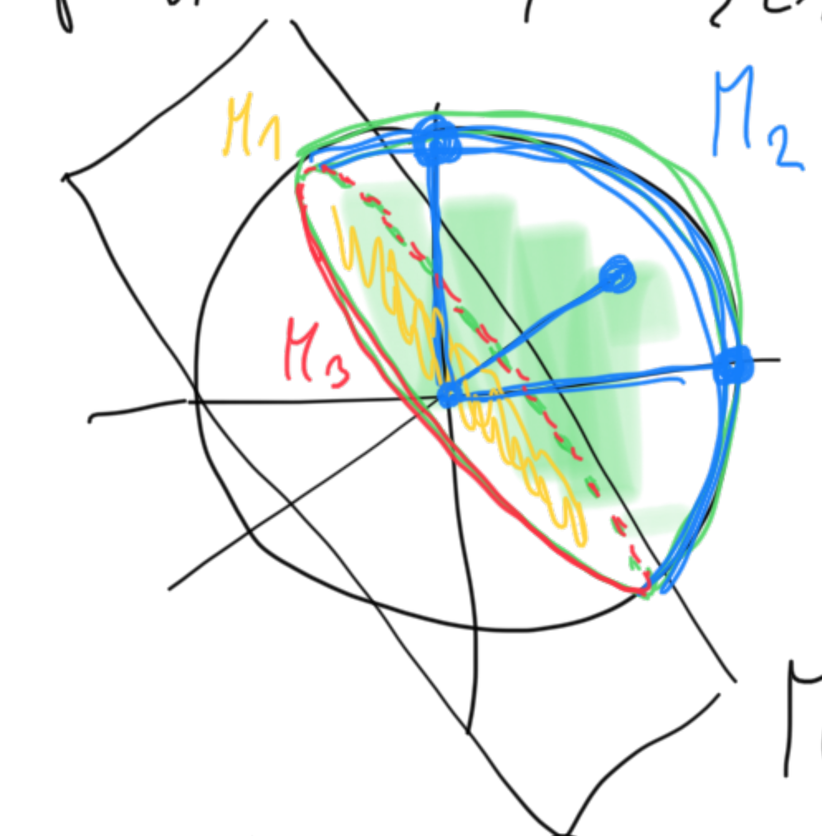
c) $x=z$

P.B. $\left[\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$, $\left[-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]$

$f = -\frac{1}{3\sqrt{6}}$ $\qquad$ $f = \frac{1}{3\sqrt{6}}$

MIN $\qquad$ MAX

$f(x,y,z) = xyz$, $M = \{[x,y,z] \in \mathbb{R}^3;\ x^2+y^2+z^2 \le 1,\ x+y+z \ge 0\}$



M je omez. (část koule o poloměru 1), M je uzavřená (průnik 2 uzav.) $\Big\}$ M je kompaktní

f je spojitá na $\mathbb{R}^3 \Rightarrow f$ nabývá na M extrémů

$M = \operatorname{Int} M \cup H(M)$

na $\operatorname{Int} M = \{x^2+y^2+z^2 < 1,\ x+y+z > 0\}$:

$$\nabla f(x,y,z) = [yz, xz, xy] = o \iff \text{aspoň 2 z } x,y,z \text{ jsou nulové}$$

Body podezřelé z extrému:
$$\boxed{\begin{array}{l} [x,0,0],\ x\in(0,1) \\ [0,y,0],\ y\in(0,1) \\ [0,0,z],\ z\in(0,1) \end{array}} \qquad f = 0$$

na $H(M) = M_1 \cup M_2 \cup M_3$

otevřená množina

$$G_2 = \{[x,y,z]\in\mathbb{R}^3;\ x^2+y^2+z^2<1\}$$

$$M_1 = \{[x,y,z]\in\mathbb{R}^3;\ x^2+y^2+z^2<1,\ \overbrace{x+y+z=0}^{g_2}\} = \{[x,y,z]\in G_2;\ g_2(x,y,z)=0\}$$

$$M_2 = \{[x,y,z]\in\mathbb{R}^3;\ \overbrace{x^2+y^2+z^2=1}^{g_1},\ x+y+z>0\} = \{[x,y,z]\in G_1;\ g_1(x,y,z)=0\}$$

$$M_3 = \{[x,y,z]\in\mathbb{R}^3;\ x^2+y^2+z^2=1,\ x+y+z=0\}$$

$$G_1 = \{x+y+z>0\}$$

otevřená

$$= \{[x,y,z]\in \underset{\mathbb{R}^3}{\underset{\shortparallel}{G}};\ g_1(x,y,z)=0,\ g_2(x,y,z)=0\}$$

<u>na M_1</u>: Lagr. věta pro 1 vazbu:

(I) $\nabla g_2 = o$

(II) $\exists \lambda\in\mathbb{R}:\ \nabla f + \lambda \nabla g_2 = o$

(I) $\nabla g_2(x,y,z) = [1,1,1] \neq \vec{o}$

(II)
$$yz + \lambda = 0$$
$$xz + \lambda = 0$$
$$xy + \lambda = 0$$
$$x+y+z = 0$$

odečtu 2. od 1.:

$z(y-x) = 0 \; < \; z=0, \; x=y$

odečtu 3. od 2.:

$x(z-y) = 0 \; < \; x=0, \; y=z$

$\underline{z=0}$ $<$ $x=0 \Rightarrow y=0$; $y=z \Rightarrow x=0$

$\underline{x=y}$ $<$ $x=0 \Rightarrow z=0$; $y=z \Rightarrow 3x=0$, $x=0$

$\Big\}$ $x=y=z=0$

P.B. $[0,0,0] \in M_1$

$\underline{\text{na } M_2}$: (I) $\nabla g_1(x,y,z) = [2x,2y,2z] = \vec{o} \iff x=y=z=0 \notin M_2$

(II)
$$yz + \lambda 2x = 0 \quad |\cdot y$$
$$xz + \lambda 2y = 0 \quad |\cdot z \quad |\cdot x$$
$$xy + \lambda 2z = 0 \quad |\cdot y$$
$$x^2+y^2+z^2 = 1$$

odečteme:

$y^2z - x^2z = 0$

$z(x^2-y^2) = 0 \; < \; z=0, \; x=\pm y$

odečteme:

$xz^2 - xy^2 = 0$

$x(z^2-y^2) = 0 \; < \; x=0$

$z=0$ — $x=0 \Rightarrow y=\pm 1$ $\quad [0,\pm 1,0]$

$y=\pm z \Rightarrow x=\pm 1$ $\quad [\pm 1,0,0]$

$x^2=y^2$ — $x=0 \Rightarrow z=\pm 1$ $\quad [0,0,\pm 1]$

$y^2=z^2 \Rightarrow 3x^2=1$

$x=\pm\frac{1}{\sqrt{3}}$ $\quad \left[\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right]$

všechny kombinace znamének

$y=\pm z$

v M_2 leží jen:

$[0,1,0]$

$[1,0,0]$

$[0,0,1]$

$f=0$

P.B.:

$\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$

$\left[-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right],\left[\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right],$
$\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right]$

na M_3 viz předchozí příklad

Dosadíme:

$$\boxed{f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\frac{1}{3\sqrt{3}}}$$

MAX

$$\boxed{f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)=-\frac{1}{3\sqrt{3}}}$$

MIN