$$6xyz - x - 2y - 3z = 5$$
$$e^{xz} = yz \qquad (*)$$

Ukažte, že existují fce $y = y(x)$, $z = z(x)$ třídy C^∞ na okolí bodu 0, které splňují $y(0) = -1$, $z(0) = -1$ a soustavu $(*)$.

Najděte $y'(0), z'(0), y''(0), z''(0)$.

Lze něco říci o průběhu fcí y, z na okolí 0?

$$F_1(x,y,z) = 6xyz - x - 2y - 3z - 5$$
$$F_2(x,y,z) = e^{xz} - yz$$

(i) $F_1 \in C^\infty(\mathbb{R}^3)$ .... polynom

$F_2 \in C^\infty(\mathbb{R}^3)$ .... aritmetika + skládání

(ii) $F_1(0,-1,-1) = 2 + 3 - 5 = 0$

$F_2(0,-1,-1) = e^0 - 1 = 0$ ✓

(iii) $\frac{\partial F_1}{\partial y}(x,y,z) = 6xz - 2 \Big|_{[0,-1,-1]} = -2$

$\frac{\partial F_1}{\partial z}(x,y,z) = 6xy - 3 \Big|_{[0,-1,-1]} = -3$

$\frac{\partial F_2}{\partial y}(x,y,z) = -z \Big|_{[0,-1,-1]} = 1$

$\frac{\partial F_2}{\partial z}(x,y,z) = e^{xz} x - y \Big| \ldots = 1$

$\partial y \quad [0,-1,-1]$ $\qquad\qquad$ $\partial z \quad \ldots = -y\big|_{[0,-1,-1]} = 1$

$$\begin{vmatrix} -2 & -3 \\ 1 & 1 \end{vmatrix} = (-2)\cdot 1 - (-3)\cdot 1 = -2+3 = 1 \neq 0$$

<u>Závěr</u>: na okolí bodu $[0,-1,-1]$ jsou soustavou $(*)$ dány jednoznačné fce $y=y(x)$ a $z=z(x)$, které jsou třídy C^∞ a splňují $y(0)=-1$, $z(0)=-1$.

Tj: $\exists$ okolí U bodu 0 takové, že $\forall x \in U$:

$$\boxed{\begin{aligned} &6x\cdot y(x)\cdot z(x) - x - 2y(x) - 3z(x) - 5 = 0, \\ &e^{x\cdot z(x)} - y(x)z(x) = 0 \end{aligned}}$$

konstantní fce, tedy mají na U nulovou derivaci

Zderivujeme:

$\forall x \in \underline{U}$:

$$6y(x)z(x) + 6x\, y'(x)\cdot z(x) + 6x\cdot y(x)\cdot z'(x) - 1 - 2y'(x) - 3z'(x) = 0$$

$$e^{x\cdot z(x)}\cdot\left(z(x) + x\cdot z'(x)\right) - y'(x)\cdot z(x) - y(x)\cdot z'(x) = 0$$

Dosadíme $x=0$, $y(0)=-1$, $z(0)=-1$:

$$6-1-2y'(0)-3z'(0)=0$$
$$-1+y'(0)+z'(0)=0 \quad /\cdot 2$$

$$5-2y'(0)-3z'(0)=0$$
$$-2+2y'(0)+2z'(0)=0$$

sečteme: $3-z'(0)=0$

$$z'(0)=3$$

$$y'(0)=-2$$

Zderivujeme ještě jednou:

$$6y'(x)z(x)+6y(x)z'(x)+6y'(x)z(x)+6x\,y''(x)\cdot z(x)+6x\,y'(x)\cdot z'(x)+6y(x)\cdot z'(x)+6x\cdot y'(x)\cdot z'(x)+$$
$$+6x\,y(x)\cdot z''(x)-2y''(x)-3z''(x)=0$$

Budu dosazovat $x=0$. Stačilo napsat:

$$6y'(x)z(x)+6y(x)z'(x)+6y'(x)z(x)+\overset{0}{x}\cdot(\dots)+6y(x)z'(x)+\overset{0}{x}\cdot(\dots)-2y''(x)-3z''(x)=0$$

$$e^{xz(x)}\cdot(z(x)+xz'(x))^2+e^{xz(x)}\cdot\left(2z'(x)+x\cdot z''(x)\right)-y''(x)z(x)-2y'(x)z'(x)-y(x)\cdot z''(x)=0$$

Dosadíme $x=0$, $y(0)=-1$, $z(0)=-1$, $y'(0)=-2$, $z'(0)=3$:

$$12(-2)(-1)+12(-1)\cdot 3-2y''(0)-3z''(0)=0$$

$$+1+6+y''(0)-2(-2)\cdot 3+z''(0)=0 \quad /\cdot 2$$

$$-12-2y''(0)-3z''(0)=0$$

$$38+2y''(0)+2z''(0)=0$$

$$26-z''(0)=0$$

$$z''(0)=26$$

$$y''(0)=-19-26=-45$$

$x \mapsto y(x) \in C^\infty$, tedy y', y'' jsou spojité $\Rightarrow$ $y'<0$ a $y''<0$ na nějakém okolí 0.

$\Rightarrow$ $x \mapsto y(x)$ je klesající a ryze konkávní na okolí 0

$x \mapsto z(x) \in C^\infty$, tedy z' a z'' jsou spojité $\Rightarrow$ $z'>0$ a $z''>0$ na nějakém okolí 0.

$\Rightarrow$ $x \mapsto z(x)$ je rostoucí a ryze konvexní na okolí 0.

$$\exp\left(\frac{u}{x}\right)\cos\left(\frac{v}{y}\right)=\frac{x}{\sqrt{2}}$$

$$\exp\left(\frac{u}{x}\right)\sin\left(\frac{v}{y}\right)=\frac{y}{\sqrt{2}}$$

Ukažte, že existují fce $[x,y]\mapsto u(x,y)$
$[x,y]\mapsto v(x,y)$

třídy C^∞ a splňující $u(1,1)=0$, $v(1,1)=\frac{\pi}{4}$

a soustavu na okolí $[1,1]$.

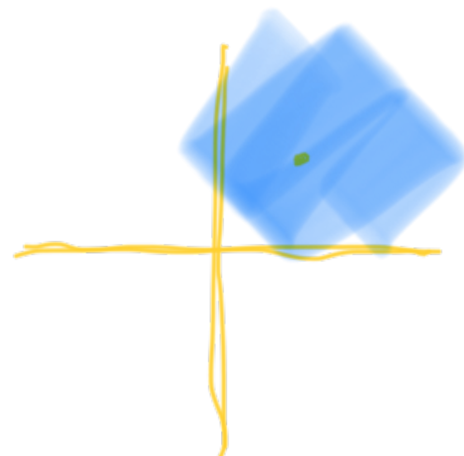
Najděte tečnou rovinu ke grafu u v bodě $[1,1,0]$, ke grafu v v bodě $[1,1,\frac{\pi}{4}]$.

$$F_1(x,y,u,v)=\exp\left(\frac{u}{x}\right)\cos\left(\frac{v}{y}\right)-\frac{x}{\sqrt{2}},$$

$$F_2(x,y,u,v)=\exp\left(\frac{u}{x}\right)\sin\left(\frac{v}{y}\right)-\frac{y}{\sqrt{2}}.$$

(i) $F_1\in C^\infty(G)$, $G=\underbrace{(0,+\infty)^2}_{x,y}\times\underbrace{\mathbb{R}^2}_{u,v}$ $\ni\left[1,1,0,\frac{\pi}{4}\right]$

$F_2\in C^\infty(G)$



$\left(G=(\mathbb{R}\setminus\{0\})^2\times\mathbb{R}^2\right)$

(ii) $F_1\left(1,1,0,\frac{\pi}{4}\right)=\exp(0)\cos\frac{\pi}{4}-\frac{1}{\sqrt{2}}=0$

$F_2\left(1,1,0,\frac{\pi}{4}\right)=\exp 0\cdot\sin\frac{\pi}{4}-\frac{1}{\sqrt{2}}=0$ ✓

(iii)

$\frac{\partial F_1}{\partial\ldots}(x,y,u,v)=\exp\left(\frac{u}{x}\right)\ldots$

$$\frac{\partial F_1}{\partial u}(x,y,u,v) = \exp\left(\frac{u}{x}\right)\cdot\frac{1}{x}\cdot\cos\frac{v}{y}\Big|_{[1,1,0,\frac{\pi}{4}]} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\frac{\partial F_1}{\partial v}(x,y,u,v) = \exp\left(\frac{u}{x}\right)\cdot\left(-\sin\frac{v}{y}\right)\cdot\frac{1}{y}\Big|_{[1,1,0,\frac{\pi}{4}]} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2},$$

$$\frac{\partial F_2}{\partial u}(x,y,u,v) = \exp\left(\frac{u}{x}\right)\cdot\frac{1}{x}\cdot\sin\frac{v}{y}\Big|_{[1,1,0,\frac{\pi}{4}]} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\frac{\partial F_2}{\partial v}(x,y,u,v) = \exp\left(\frac{u}{x}\right)\cos\frac{v}{y}\cdot\frac{1}{y}\Big|_{[1,1,0,\frac{\pi}{4}]} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

$$\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}+\frac{1}{2} = 1 \neq 0 \quad \checkmark$$

Existují tedy okolí U bodu $[1,1]$ a okolí V bodu $[0,\frac{\pi}{4}]$ tak, že

$$\forall [x,y] \in U \quad \exists!\ [u(x,y), v(x,y)] \in V:$$

$$F_1(x,y,u(x,y),v(x,y)) = 0 \ \& \ F_2(x,y,u(x,y),v(x,y)) = 0,$$

přičemž u, v jsou třídy C^∞ na U a $u(1,1)=0$, $v(1,1)=\frac{\pi}{4}$.

Pro $[x,y]\in U$ platí:

$$\boxed{\begin{aligned} &\exp\left(\frac{u(x,y)}{x}\right)\cdot\cos\frac{v(x,y)}{y}-\frac{x}{\sqrt2}=0,\\ &\exp\left(\frac{u(x,y)}{x}\right)\cdot\sin\frac{v(x,y)}{y}-\frac{y}{\sqrt2}=0.\end{aligned}}$$

konst. fce 2 proměnných (x,y)

Parc. der. dle x:

$$\exp\left(\frac{u(x,y)}{x}\right)\cdot\frac{\frac{\partial u}{\partial x}(x,y)\cdot x-1\cdot u(x,y)}{x^2}\cdot\cos\frac{v(x,y)}{y}\;-$$

$$-\exp\left(\frac{u(x,y)}{x}\right)\cdot\sin\left(\frac{v(x,y)}{y}\right)\cdot\frac{1}{y}\cdot\frac{\partial v}{\partial x}(x,y)-\frac{1}{\sqrt2}=0$$

$$\exp\left(\frac{u(x,y)}{x}\right)\cdot\frac{\frac{\partial u}{\partial x}(x,y)\,x-u(x,y)}{x^2}\cdot\sin\frac{v(x,y)}{y}\;+$$

$$+\exp\left(\frac{u(x,y)}{x}\right)\cdot\cos\left(\frac{v(x,y)}{y}\right)\cdot\frac{1}{y}\cdot\frac{\partial v}{\partial x}(x,y)=0$$

Dosadíme $x=1,\ y=1,\ u(1,1)=0,\ v(1,1)=\frac{\pi}{4}$:

$$\frac{\partial u}{\partial x}(1,1)\cdot\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cdot\frac{\partial v}{\partial x}(1,1)-\frac{1}{\sqrt2}=0$$

$$\frac{\partial u}{\partial x}(1,1)\cdot \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cdot\frac{\partial v}{\partial x}(1,1) = 0$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}(1,1) - \frac{\partial v}{\partial x}(1,1) &= 1\\ \frac{\partial u}{\partial x}(1,1) + \frac{\partial v}{\partial x}(1,1) &= 0\end{aligned}\right\} \Rightarrow \boxed{\begin{aligned}\frac{\partial u}{\partial x}(1,1) &= \frac{1}{2}\\ \frac{\partial v}{\partial x}(1,1) &= -\frac{1}{2}\end{aligned}}$$

$\vdots$

$$\frac{\partial u}{\partial y}(1,1) = \frac{1}{2}, \quad \frac{\partial u}{\partial y}(1,1) = \frac{\pi}{4} + \frac{1}{2}$$

Tečná rovina k u: $T(x,y) = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

k v: $T(x,y) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \left(\frac{\pi}{4} + \frac{1}{2}\right)(y-1)$.