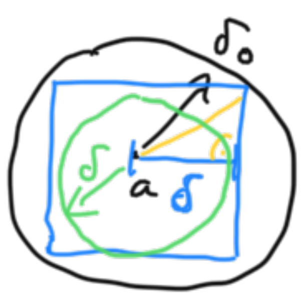
Důkaz: Zvolme $\varepsilon > 0$.

Protože f má v a všechny parc. der. spojité, tedy

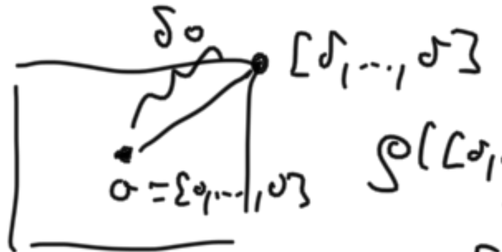
$\forall i \in \{1,\dots,n\}$ existuje $\delta_i > 0$ takové, že $\left|\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a)\right| < \frac{\varepsilon}{n}$ kdykoli $x \in B(a,\delta_i)$.

Položíme $\delta_0 = \min\{\delta_1,\dots,\delta_n\}$ a $\delta = \frac{\delta_0}{\sqrt{n}}$.



$n=2: \ \delta = \frac{\delta_0}{\sqrt{2}}$

$n=3: \ \delta = \frac{\delta_0}{\sqrt{3}}$



$\rho([\delta,\delta,\dots,\delta],0) = \delta_0 = \sqrt{n}\cdot\delta$

Označme $I = (a_1-\delta, a_1+\delta) \times \dots \times (a_n-\delta, a_n+\delta)$.

Pak $I \subset B(a,\delta_0)$.

Tedy $\forall x \in I: \left|\frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(a)\right| < \frac{\varepsilon}{n}$.

Zvolme libovolné $y \in B(a,\delta) \subset I$.

Pak $y \in I$, dle T18 (zl. Lagr. v.) $\exists\, \xi^1,\dots,\xi^n \in I$ takové, že

$$f(y) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\xi^i)\cdot(y_i - a_i).$$

Tudíž
$$\left|\frac{f(y)-T(y)}{\rho(y,a)}\right| = \left|f(y) - f(a) - \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)\cdot(y_i-a_i)\right| \le$$

$$= \frac{\left|\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}(\xi^i) - \frac{\partial f}{\partial x_i}(a)\right)\cdot(y_i-a_i)\right|}{\rho(y,a)} \underset{\text{△ nerovnost}}{\leq} \frac{\sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}(\xi^i) - \frac{\partial f}{\partial x_i}(a)\right|\cdot|y_i-a_i|}{\rho(y,a)} \leq$$

$$\leq \frac{\sum_{i=1}^{n}\frac{\varepsilon}{n}|y_i-a_i|}{\rho(y,a)} \leq \frac{\varepsilon}{n}\cdot\frac{\sum_{i=1}^{n}\rho(y,a)}{\rho(y,a)} = \frac{\varepsilon}{n}\frac{n\cdot\rho(y,a)}{\rho(y,a)} = \varepsilon. \qquad \square$$

$n=1$ $\quad F(x)=f(\varphi(x))$, $\quad b=\varphi(a)$

$F'(a)=f'(b)\cdot\varphi'(a)$

$f(\vec{y}) = f(y_1,\dots,y_r)$

$\frac{\partial f}{\partial y_i}$ ... parc. der. fce f podle její i-té proměnné

$$\frac{\partial F}{\partial x_j}(a)=\sum_{i=1}^{r}\frac{\partial f}{\partial y_i}(b)\cdot\frac{\partial \varphi_i}{\partial x_j}(a) \quad \text{..... řetízkové pravidlo}$$

$$F(x)=f(\underset{y_1}{\varphi_1(x)},\dots,\underset{y_i}{\varphi_i(x)},\dots,\underset{y_r}{\varphi_r(x)})$$

$r=s=2$

$f(y_1,y_2)=y_1+y_2$

$\varphi_1(x_1,x_2)=x_2^2$

$\varphi_2(x_1,x_2) = x_1 \cdot x_2$

$$F(x_1,x_2) = f(\varphi_1(x_1,x_2), \varphi_2(x_1,x_2)) = \varphi_1(x_1,x_2) + \varphi_2(x_1,x_2) = x_2^2 + x_1 \cdot x_2$$

$$\frac{\partial F}{\partial x_1}(2,2) = \frac{\partial f}{\partial y_1}(4,4) \cdot \frac{\partial \varphi_1}{\partial x_1}(2,2) + \frac{\partial f}{\partial y_2}(4,4) \cdot \frac{\partial \varphi_2}{\partial x_1}(2,2) = 1 \cdot 0 + 1 \cdot 2 = 2$$

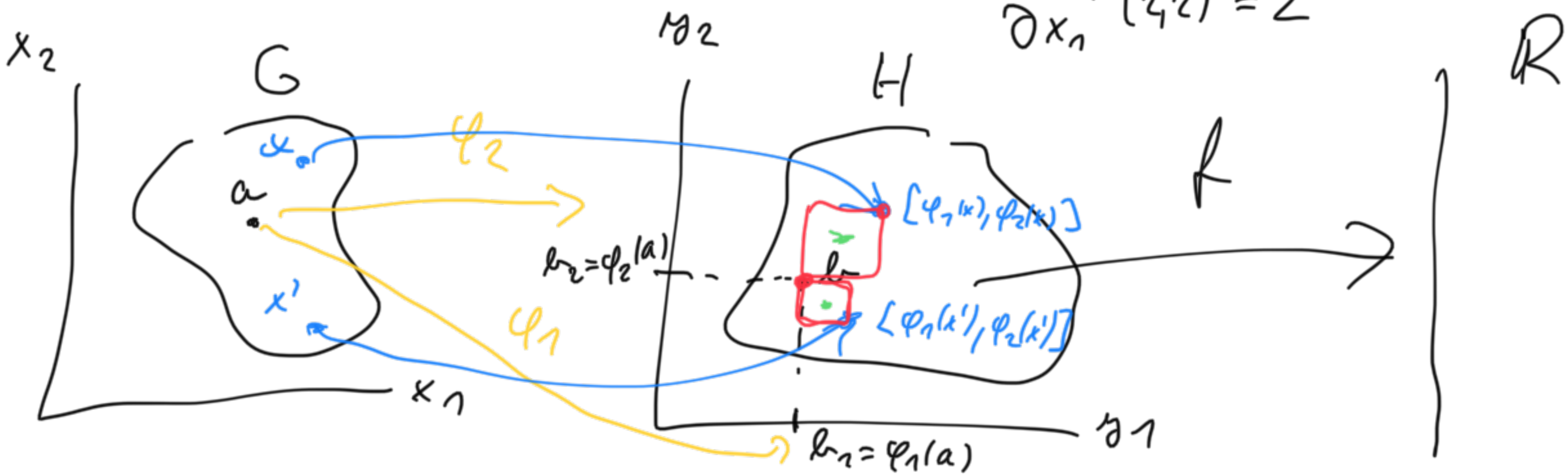
$a = [2,2]$, $\varphi_1(2,2) = 4$, $\varphi_2(2,2) = 4$

$b = [4,4]$

$$\frac{\partial f}{\partial y_1}(4,4) = 1, \quad \frac{\partial f}{\partial y_2}(4,4) = 1$$

$$\frac{\partial \varphi_1}{\partial x_1}(2,2) = 0, \quad \frac{\partial \varphi_2}{\partial x_1}(x_1,x_2) = x_2$$

$$\frac{\partial \varphi_2}{\partial x_1}(2,2) = 2$$



Důkaz: Stačí dokázat vzoreček a použít aritmetiku spoj. fcí $\Rightarrow F \in C^1(G)$.

BÚNO $s=1$ (bereme v úvahu jen přísl. řezy fcí $\varphi_1, \dots, \varphi_r, F$)

Chceme vypočítat $F'(a) = \lim_{x \to a} \frac{F(x)-F(a)}{x-a}$.

H je otevřená, tedy $\exists\, \Delta > 0 : B(b, \Delta) \subset H$.

Položím $\varepsilon = \frac{\Delta}{\sqrt{n}}$, $I_i = (b_i - \varepsilon, b_i + \varepsilon)$, $i = 1, \dots, n$.

Pak $I = I_1 \times \dots \times I_n \subset B(b, \Delta) \subset H$.

G je otevřená a $\varphi_1, \dots, \varphi_n$ jsou spojité (V20).

Tedy $\exists\, \delta_i > 0 : (a - \delta_i, a + \delta_i) \subset G$ a $\forall x \in (a - \delta_i, a + \delta_i) : \varphi_i(x) \in I_i$.

Položím $\delta = \min\{\delta_1, \dots, \delta_n\}$.

Pak $\forall x \in (a - \delta, a + \delta)$ je $\varphi_i(x) \in I_i$, $i = 1, \dots, n$.

Nechť $x \in (a - \delta, a + \delta)$. Pak

$$F(x) - F(a) = f(\varphi_1(x), \dots, \varphi_n(x)) - f(\varphi_1(a), \dots, \varphi_n(a)) =$$

$$= f(\underbrace{\varphi_1(x), \dots, \varphi_n(x)}_{\in I}) - f(\underbrace{b_1, \dots, b_n}_{\in I}).$$

Použijeme T18 (vl. Lagr. v.) na fci f, I a body $[\varphi_1(x), \dots, \varphi_n(x)]$, $[b_1, \dots, b_n]$:

Existují body $\xi^1(x), \ldots, \xi^n(x) \in I$, že

POZOR! body ξ^i závisejí na x

$$\xi^i(x)_j \in \langle b_j, \varphi_j(x) \rangle, \quad i, j \in \{1, \ldots, n\}$$

$b_j \leq \xi^i(x)_j \leq \varphi_j(x)$

$\downarrow$

b_j

a $$f(\varphi_1(x), \ldots, \varphi_n(x)) - f(b_1, \ldots, b_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(\xi^i(x)) \cdot (\varphi_i(x) - b_i)$$

Ze spojitosti φ_i v bodě a plyne, že

$$\lim_{x \to a} \varphi_i(x) = \varphi_i(a) = b_i.$$

Dle věty o 2 policajtech je tedy $\lim_{x \to a} \xi^i(x)_j = b_j$.

Fce $\frac{\partial f}{\partial y_i}$ jsou spojité v b, takže dle VOLSF o spojn. (S) (V16)

je tedy $$\lim_{x \to a} \frac{\partial f}{\partial y_i}(\underbrace{\xi^i(x)}_{\to b}) = \frac{\partial f}{\partial y_i}(b).$$

Dokončeme:

$$F'(a) = \lim \frac{F(x) - F(a)}{\ldots} = \lim \frac{\sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(\xi^i(x)) \cdot (\varphi_i(x) - b_i)}{\ldots} =$$

$$x \to a \qquad x-a \qquad x \to a \qquad x-a$$

$$= \lim_{x \to a} \sum_{i=1}^{k} \underbrace{\frac{\partial f}{\partial y_i}\left(\xi^i(x)\right)}_{\to \frac{\partial f}{\partial y_i}(b)} \cdot \underbrace{\frac{\varphi_i(x) - \varphi_i(a)}{x-a}}_{\to \varphi_i'(a)} \overset{AL}{=} \sum_{i=1}^{k} \frac{\partial f}{\partial y_i}(b) \cdot \varphi_i'(a).$$

$\square$