

Very transitive groups and geometries

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Definition

Let G act on X .

- A subset $Orb(x) = \{g(x) \in X \mid g \in G\}$ is called *orbit of point x* .
- A subgroup $Stab(x) = \{g \in G \mid g(x) = x\}$ is called *stabilizer of point x* .

Theorem (orbit-stabilizer property)

Let G act on X . For all $x \in X$ we have $|Orb(x)| = |G : Stab(x)|$ (or $|Orb(x)||Stab(x)| = |G|$ if G is finite).

Sketch of proof.

$\phi : gStab(x) \mapsto g(x)$ is a bijection between cosets and orbit elements.
Injectivity / correctness is concluded from

$$g(x) = h(x) \Leftrightarrow h^{-1}g(x) = x \Leftrightarrow h^{-1}g \in Stab(x) \Leftrightarrow gStab(x) = hStab(x)$$



Definition

Let G act on X . The actions is

- *transitive*, if for all $x, y \in X$ there is $g \in G$ such that $g(x) = y$.
- *k-transitive*, if for all $x_1 \neq \dots \neq x_k, y_1 \neq \dots \neq y_k \in X$ there is $g \in G$ such that $g(x_j) = y_j$ for all $1 \leq j \leq k$.

Example

For every action of G on X we can define an action of G on the set of distinct k -tuples $X^{(k)}$ by

$$\mathbf{x} = (x_1, \dots, x_k) \mapsto (g(x_1), \dots, g(x_k)) \in X^{(k)}.$$

Remark

Let G act on X .

- G acts transitively, iff G has one orbit on X .
- k -transitivity implies $(k - 1)$ -transitivity.
- G is k -transitive on X , iff it is transitive on $X^{(k)}$.

Theorem

Let G act k -transitively on X and $|X| = n$. Then $n(n-1)\dots(n-k+1)$ divides $|G|$.

Proof.

By remark and orbit-stabilizer property we have for every $\mathbf{x} \in X^{(k)}$:

$$|G| = |\text{Orb}(\mathbf{x})| |\text{St}(\mathbf{x})| = |X^{(k)}| |\text{St}(\mathbf{x})| = n(n-1)\dots(n-k+1) |\text{St}(\mathbf{x})|$$



Example

- S_n is n -transitive on $\{1, \dots, n\}$.
- A_n is $(n-2)$ -transitive on $\{1, \dots, n\}$.
- D_n is 1-transitive on $\{1, \dots, n\}$.

Lemma

Let G act on X . The action is k -transitive, iff:

- (1) G acts transitively on X .
- (2) Every point stabilizer $\text{Stab}(x)$ acts $(k - 1)$ -transitively on $X \setminus \{x\}$.

The condition (2) can be equivalently replaced with

- (2') Some point stabilizer $\text{Stab}(y)$ acts $(k - 1)$ -transitively on $X \setminus \{y\}$.

Sketch of proof.

- k -transitivity \Rightarrow (1): Remark
- k -transitivity \Rightarrow (2): find such g for every choice of x_i, y_i :
 $g(x_1, x_2, \dots, x_{k-1}, x) = (y_1, y_2, \dots, y_{k-1}, x)$
- (1) + (2) \Rightarrow k -transitivity: find hg , such that

$$g(x_1, \dots, x_k) = (z_1, \dots, z_{k-1}, y_k)$$
$$h(z_1, \dots, z_{k-1}, y_k) = (y_1, \dots, y_{k-1}, y_k)$$



Example

Let $X = (V, E)$ be a graph. If $\text{Aut}(X)$ is 2-transitive, then $X = K_n$ or $X = \bar{K}_n$.

Example

$GL_n(\mathbb{F}_q)$ acts transitively on $\mathbb{F}_q \setminus \{0\}$. The action is 2-transitive only in cases

- 1 $q = 3$ a $n = 1$,
- 2 $q = 2$ a $n > 1$.

Example

$AGL_n(\mathbb{F}_q)$ acts 2-transitively on \mathbb{F}_q . The action is 3-transitive only in cases from previous example.

Example

$PGL_n(\mathbb{F}_q)$ acts 2-transitively on projective space and acts 3-transitively, iff $n = 2$.

Definition

A structure $S(\Omega, \mathcal{B})$, where Ω is a finite set and \mathcal{B} is a system of subsets (blocks), is called a *Steiner system* if

- 1 all blocks have the same size k ,
- 2 for some $t \in \mathbb{N}$, every subset of size t lies in exactly one block.

Such Steiner system has parameters $S(t, k, v)$, where $v = |\Omega|$.

Example

Affine space over \mathbb{F}_q is a $S(2, q, q^d)$ Steiner system for $q > 2$. Its automorphism group is $A\Gamma L_d(\mathbb{F}_q)$, which equals $AGL_d(\mathbb{F}_q)$ for q prime.

Example

Projective space over \mathbb{F}_q is a $S(2, q + 1, (q^d - 1)/(q - 1))$ Steiner system. Its automorphism group is $P\Gamma L_{d+1}(\mathbb{F}_q)$, which equals $PGL_{d+1}(\mathbb{F}_q)$ for q prime.

Theorem

Let $S(\Omega, \mathcal{B})$ be a $S(t, k, v)$ Steiner system.

- 1 There are $r = \binom{v-1}{t-1} / \binom{k-1}{t-1}$ blocks containing a particular point.
- 2 There are $b = \frac{vr}{k}$ blocks.

Sketch of proof.

- 1 From definition, t items uniquely determines a block. For a fixed point, there are $\binom{v-1}{t-1}$ ways to pick the other $t-1$ points. However, $\binom{v-1}{t-1}$ of them are from the same block.
- 2 Count the elements of $M = \{(\alpha, B) \mid B \in \mathcal{B}, \alpha \in B\}$ in two ways.



Remark

It's possible to show that there exist unique Steiner systems of parameters denoted in the table below. Groups M_{11} , M_{12} , M_{23} and M_{24} are constructed as their automorphism groups.

Steiner system	Parameters	Blocks	Automorphism group
W_{11}	$(4,5,11)$	66	M_{11}
W_{12}	$(5,6,12)$	132	M_{12}
W_{23}	$(4,7,23)$	253	M_{23}
W_{24}	$(5,8,24)$	759	M_{24}

Table: Mathieu groups

Theorem

Let G act faithfully k -transitively on X with $k \geq 4$, where X is finite. Then either $G \cong S_n$ (for $n \geq 4$), $G \cong A_n$ ($n \geq 6$) or $G \cong M_n$ ($n \in \{11, 12, 23, 24\}$).

Bonus - Geometry of W_{12}

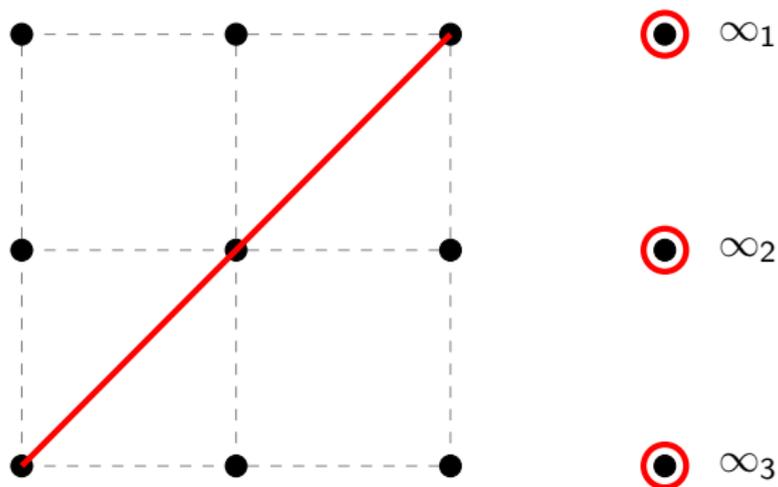


Figure: Line in W_{12}

Bonus - Geometry of W_{12}

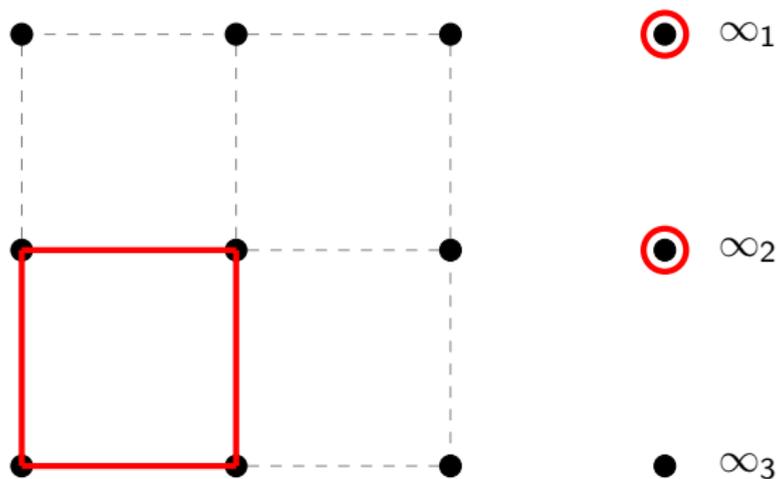


Figure: Quadrangle in W_{12}

Bonus - Geometry of W_{12}

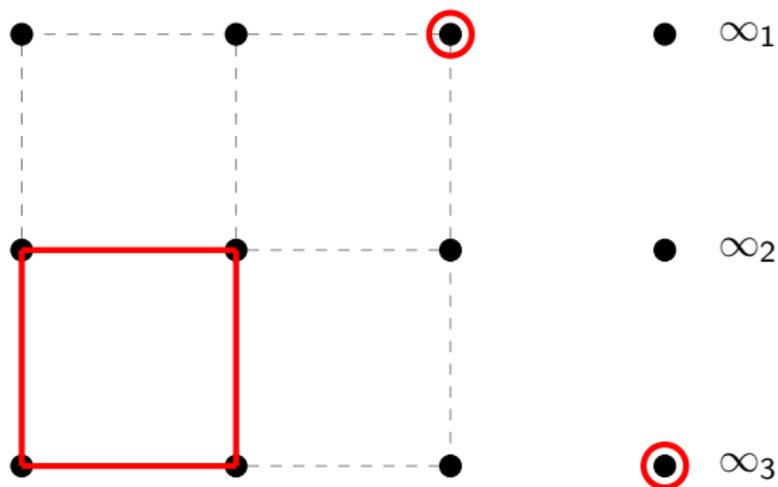


Figure: Quadrangle with a diagonal point in W_{12}

Bonus - Geometry of W_{12}

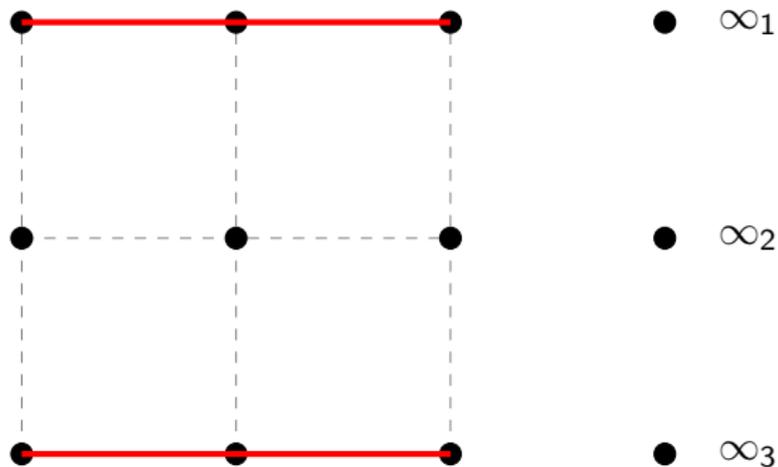


Figure: Parallel lines in W_{12}