QUASI-EQUATIONAL AXIOMATIZATIONS FOR GRAPHS OF SEMIGROUPS, MONOIDS AND GROUPS

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The graph of an algebra $A = (A, \Omega)$ is the relational structure
$$G(A) = (A, \{R_\omega \}_{\omega \in \Omega}),$$
where each $R_\omega$ is the graph of an operation $\omega$. This means that if $\omega$ is an $n$-ary operation, then $R_\omega$ is the $(n+1)$-ary relation consisting of those tuples $(a_1, \ldots, a_{n+1})$ which satisfies $\omega(a_1, \ldots, a_n) = a_{n+1}$. For a class $C$ of algebras by $G(C)$ we denote the class of all graphs of algebras from $C$.

A quasiversity is a class defined by quasi-identities, i.e. by sentences of the form
$$(\forall \bar{x}) \left[ \varphi_1(\bar{x}) \land \cdots \land \varphi_n(\bar{x}) \rightarrow \varphi(\bar{x}) \right],$$
where $n$ is a natural number and $\varphi_1, \ldots, \varphi_n, \varphi$ are atomic formulas. For a class $K$ by $Q(K)$ we denote the smallest quasiversity containing $K$. It is the class defined by the quasi-identities true in $K$.

O. M. Gornostaev proved that $QG(\mathbb{Z}_2, +)$ and $QG(\mathbb{Z}_2, \cdot)$ are not finitely axiomatizable. This sharply contrasts with the facts that all varieties and quasivarieties generated by two element algebras are finitely axiomatizable. Our motivation was to check whether Gornostaev’s result is just a curious exception or there is a deeper reason for the lack of a finite quasi-equational basis. We obtained the following fact.

Theorem. Let $C$ be a class of semigroups possessing a nontrivial member with a neutral element. Then $QG(C)$ is not finitely axiomatizable.

Small modifications in the proof yield the following result.

Corollary. Let $C$ be a class of monoids or groups possessing a nontrivial member. Then $QG(C)$ is not finitely axiomatizable.

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