Lattice Universal Semigroup Varieties

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\( \Pi_\infty \) is universal in the following sense: it contains the congruence lattice of every countable algebra as a sublattice. 
In particular, \( \Pi_\infty \) contains the dual of the lattice of all varieties of algebras of any fixed finite or countably infinite type \( \tau \) since the latter lattice is the dual of the lattice of fully invariant congruences on the free countably generated algebra of type \( \tau \).
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\[
L(\mathcal{V})^d \cong \text{Con}_{fi}(F_\infty(\mathcal{V})) \hookrightarrow \text{Con}(F_\infty(\mathcal{V})) \hookrightarrow \Pi_\infty
\]
A variety of semigroups is called **lattice universal** if its subvariety lattice contains an interval dual to $\Pi_\infty$. 
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If $\mathcal{U}$ is a lattice universal semigroup variety, then $L(\mathcal{U})$ contains an isomorphic copy of $L(\mathcal{V})$ for **every** variety $\mathcal{V}$ of algebras of any finite or countably infinite type $\tau$. 
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For Ježek’s result, the mere existence of such a sequence wasn’t sufficient; he needed an infinite sequence of square-free words with an additional combinatorial property.
Let $\Sigma$ and $\Delta$ be two alphabets (which may be equal), and let $\Sigma^+$ and $\Delta^+$ stand for the sets of non-empty words over $\Sigma$ and $\Delta$ respectively.
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**Definition**

A word $\nu \in \Sigma^+$ encounters a word $p \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(p)$ is a factor of $\nu$. 
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A word $\nu \in \Sigma^+$ encounters a word $\rho \in \Delta^+$ if there is a map $h : \Delta \to \Sigma^+$ such that $h(\rho)$ is a factor of $\nu$.

- The word *banana* encounters the word $x^2$. 
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- The word $ba \ na \ na$ encounters the word $x^2$. 

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- The word $ba na na$ encounters the word $x^2$.

Otherwise the word $\nu$ is said to **avoid** the word $p$. 

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Otherwise the word $\nu$ is said to avoid the word $p$.

In particular, a word is square-free iff it avoids $x^2$. 
Antichains

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Ježek constructed such an antichain. He wrote that Thue’s paper was not accessible to him so he did not know whether or not Thue already had this result. In fact, Thue had several constructions but none of them yield Ježek’s result.
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- $x^2 \cong 0$, 

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The rest is easy. Let $A$ be an infinite antichain consisting of square-free words on three symbols. Given a partition $\pi$ of $A$, we define a semigroup variety $\mathcal{V}_\pi$ by the following identities:

- $x^2 \simeq 0$,
- all identities $w \simeq 0$ where $w$ encounters some word in $A$,
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- $x^2 \simeq 0$,
- all identities $w \simeq 0$ where $w$ encounters some word in $A$,
- all identities $u \simeq v$ such that $u, v \in A$ lie in the same $\pi$-class.

Then $\pi \mapsto \mathcal{V}_\pi$ is a dual isomorphism between $\Pi_\infty$ and an interval in the subvariety lattice of the variety $\mathcal{J}^2$ defined by $x^2 \simeq 0$. 

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Thus, \( J^2 \) is lattice universal and so is every variety containing \( J^2 \). Are there any further examples? Is it possible to somehow classify lattice universal varieties? Given a finite set of identities, can one decide if these identities define a lattice universal variety? Addressing these question has become possible only after a crucial progress in combinatorics on words was achieved by Bean, Ehrenfeucht, McNulty (1979) and Zimin (1982).
A word $w$ is called **avoidable** if there exists an infinite sequence of words \{${u_i}$\} over a finite alphabet such that each word $u_i$ avoids $w$. 
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Zimin’s words $Z_n \in \{x_1, x_2, \ldots\}^+$ are defined as follows:

\[ Z_1 = x_1, \]
\[ Z_2 = x_1 x_2 x_1, \]
\[ \ldots \]
\[ Z_{n+1} = Z_n x_{n+1} Z_n, \]
\[ \ldots \]
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$$\ldots$$

$$Z_{n+1} = Z_nx_{n+1}Z_n,$$

$$\ldots$$

**Theorem (Zimin, 1982)**

A word $w$ involving $n$ different letters is unavoidable iff the word $Z_n$ encounters $w$. 
It is relatively easy to extend Ježek’s theorem to any variety defined by an identity of the form $w \equiv 0$ where $w$ is an avoidable word or even by a system of identities $w_i \equiv 0$ where all $w_i$ are avoidable and involve only finitely many letters.
It is relatively easy to extend Ježek’s theorem to any variety defined by an identity of the form \( w \cong 0 \) where \( w \) is an avoidable word or even by a system of identities \( w_i \cong 0 \) where all \( w_i \) are avoidable and involve only finitely many letters.

For this, one should only show that for every avoidable word \( w \), there exists an infinite antichain \( A_w \) such that all words in \( A_w \) involve only finitely many letters and avoid \( w \); then Ježek’s argument applies.
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A construction for such an antichain \( A_w \) was found by Mikhailova in 2009.
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**Example**

The variety $\mathcal{M}_3$ defined by the identity

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq bxa \cdot y \cdot axb \cdot z \cdot axb \cdot y \cdot bxa$$

is lattice universal while its subvariety defined by

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \simeq 0$$

is locally finite, and therefore, is not lattice universal.
Counter-Example

This generalization is not yet sufficient:

Example

The variety $\mathcal{M}_3$ defined by the identity

$$axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb \cong bxa \cdot y \cdot axb \cdot z \cdot axb \cdot y \cdot bxa$$

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The local finiteness of the subvariety follows from the fact that the word $axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb$ is unavoidable.
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The local finiteness of the subvariety follows from the fact that the word $axb \cdot y \cdot bxa \cdot z \cdot bxa \cdot y \cdot axb$ is unavoidable. The subvariety lattice of a locally finite variety is algebraic and hence it cannot contain $\Pi_\infty$ as an interval.
Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most $n$ letters and satisfies no non-trivial identity of the form $\mathbb{Z}_{n+1} \cong \mathbb{Z}$. Then $\mathcal{V}$ is lattice universal.
Further Generalization

Theorem 1

Suppose that a semigroup variety $\mathcal{V}$ is defined by identities depending on at most $n$ letters and satisfies no non-trivial identity of the form $Z_{n+1} \cong w$. Then $\mathcal{V}$ is lattice universal.

In particular, the lattice universality of $\mathcal{M}_3$ follows as soon as one verifies that $\mathcal{M}_3$ satisfies no non-trivial identity of the form $Z_6 \cong w$. 
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Our proof of Theorem 1 relies on constructing an infinite antichain $A_\mathcal{V}$ such that $\mathcal{V}$ satisfies no non-trivial identity of the form $u \cong v$ with $u \in A_\mathcal{V}$.
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Our proof of Theorem 1 relies on constructing an infinite antichain $A_\mathcal{V}$ such that $\mathcal{V}$ satisfies no non-trivial identity of the form $u \cong v$ with $u \in A_\mathcal{V}$. Then Ježek’s argument applies mutatis mutandis.
Let $r' = \lceil \log_2(6n - 1) \rceil$ and $r = 2^{r'}$. 
Let $r' = \lceil \log_2(6n - 1) \rceil$ and $r = 2^{r'}$. Consider $r^2 \times r$ matrix

$$
P = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & r & \ldots & 1 & r \\
2 & 1 & \ldots & 2 & 1 \\
2 & r & \ldots & 2 & r \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 r & 1 & \ldots & r & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 r & r & \ldots & r & r \\
\end{pmatrix}
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2 & 1 & \cdots & 2 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
r & 1 & \cdots & r & 1 \\
r & r & \cdots & r & r
\end{pmatrix}
$$

$$
P_{\Sigma} = \begin{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1r-1} & a_{1r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{11} & a_{r2} & \cdots & a_{1r-1} & a_{rr} \\
a_{21} & a_{12} & \cdots & a_{2r-1} & a_{1r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{21} & a_{r2} & \cdots & a_{2r-1} & a_{rr} \\
a_{r1} & a_{12} & \cdots & a_{rr-1} & a_{1r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{r1} & a_{r2} & \cdots & a_{rr-1} & a_{rr}
\end{pmatrix}
\end{pmatrix}
$$

Whenever $i$ occurs in the column $j$, we substitute $i$ by $a_{ij}$. 
Let $r' = \lfloor \log_2(6n - 1) \rfloor$ and $r = 2^{r'}$. Consider $r^2 \times r$ matrix

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1 & 1 & \cdots & 1 & 1 \\
1 & r & \cdots & 1 & r \\
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r & r & \cdots & r & r \\
\end{pmatrix}
$$

$$
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\end{array}
\end{pmatrix}
$$

Whenever $i$ occurs in the column $j$, we substitute $i$ by $a_{ij}$.

$$
\Sigma = \{a_{ij} \mid 1 \leq i, j \leq r\}.
$$
Let $v_i$ be the word in $i$-th row of the matrix $P_\Sigma$. 
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Let $\nu_i$ be the word in $i$-th row of the matrix $P_\Sigma$. We define the map $\gamma : \Sigma \to \Sigma^+$ by $\gamma(a_{ij}) = \nu(i-1)r+j$.

**Theorem (Sapir, 1987)**

Each word of the sequence $\gamma^m(a_{11})$ avoids all avoidable words over $n$ letters.
Let \( v_i \) be the word in \( i \)-th row of the matrix \( P_\Sigma \). We define the map \( \gamma : \Sigma \to \Sigma^+ \) by \( \gamma(a_{ij}) = v_{(i-1)r+j} \).

**Theorem (Sapir, 1987)**

Each word of the sequence \( \gamma^m(a_{11}) \) avoids all avoidable words over \( n \) letters.

If \( w = x_1 x_2 \ldots x_\ell \) is a word, we denote by \( \overleftarrow{w} = x_\ell \ldots x_2 x_1 \) the mirror image of \( w \). A word \( w \) is a palindrome if \( \overleftarrow{w} = w \).
Theorem (Mikhaylova, Volkov, 2007)

Take $d_1 \notin \Sigma$. Then each word in the sequence of palindromes

$$\sigma_m = \gamma^m(a_{11})d_1\gamma^m(a_{11})$$

avoids all avoidable words over $n$ letters.
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Take $d_2 \notin \{d_1\} \cup \Sigma$. Then the sequence $\eta_m = d_2 \sigma_m d_2 d_1$ where $m \geq r^2$ is our antichain $A_\gamma$. 
Theorem 2

Suppose that a lattice universal semigroup variety \( \mathcal{V} \) is defined by identities depending on at most \( n \) letters and all periodic groups in \( \mathcal{V} \) are locally finite. Then \( \mathcal{V} \) satisfies no non-trivial identity of the form \( \mathbb{Z}_{n+1} \cong \mathfrak{w} \).
Theorem 2

Suppose that a lattice universal semigroup variety $\mathcal{V}$ is defined by identities depending on at most $n$ letters and all periodic groups in $\mathcal{V}$ are locally finite. Then $\mathcal{V}$ satisfies no non-trivial identity of the form $\mathbb{Z}_{n+1} \cong \nu$.

In other words, we have obtained a complete characterization of lattice universal semigroup varieties in the class of varieties defined by identities in finitely many letters and containing no infinite periodic groups with finitely many generators.
Theorem 2

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In other words, we have obtained a complete characterization of lattice universal semigroup varieties in the class of varieties defined by identities in finitely many letters and containing no infinite periodic groups with finitely many generators. Given a finite set of identities that force periodic groups to be locally finite, we can effectively check whether or not these identities define a lattice universal variety.
Embedding \((\Pi_\infty)^d\) into the Lattice of Semigroup Varieties

Overcommutative varieties

Periodic varieties
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Periodic varieties
Thus, if \((\Pi_{\infty})^d\) embeds into \(L(\mathcal{V})\) for some semigroup variety \(\mathcal{V}\), it must embed into \(L(\mathcal{P})\) where \(\mathcal{P}\) is a periodic subvariety of \(\mathcal{V}\).
Thus, if \((\Pi_\infty)^d\) embeds into \(L(\mathcal{V})\) for some semigroup variety \(\mathcal{V}\), it must embed into \(L(\mathcal{P})\) where \(\mathcal{P}\) is a periodic subvariety of \(\mathcal{V}\).

**Theorem (Sapir, 1987)**

Suppose that a periodic semigroup variety \(\mathcal{P}\) is defined by identities depending on at most \(n\) letters and all groups in \(\mathcal{P}\) are locally finite. Then \(\mathcal{P}\) is locally finite iff \(\mathcal{P}\) satisfies a non-trivial identity of the form \(Z_{n+1} \simeq w\).
Thus, if \((\Pi_\infty)^d\) embeds into \(L(\mathcal{V})\) for some semigroup variety \(\mathcal{V}\), it must embed into \(L(\mathcal{P})\) where \(\mathcal{P}\) is a periodic subvariety of \(\mathcal{V}\).

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We have already seen that a lattice universal variety cannot be locally finite.
Thus, if \((\Pi_\infty)^d\) embeds into \(L(\mathcal{V})\) for some semigroup variety \(\mathcal{V}\), it must embed into \(L(\mathcal{P})\) where \(\mathcal{P}\) is a periodic subvariety of \(\mathcal{V}\).

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We have already seen that a lattice universal variety cannot be locally finite. Thus, \(\mathcal{P}\) (and hence \(\mathcal{V}\)) satisfies no non-trivial identity of the form \(Z_{n+1} \cong w\).
Future Work

A semigroup variety $\mathcal{V}$ is said to be finitely universal if for each $n$, the lattice $L(\mathcal{V})$ contains an interval dual to $\Pi_n$, the partition lattice on an $n$-element set.
A semigroup variety $\mathcal{V}$ is said to be **finitely universal** if for each $n$, the lattice $L(\mathcal{V})$ contains an interval dual to $\Pi_n$, the partition lattice on an $n$-element set. Examples: the variety of all commutative semigroups, the variety defined by $x^2 \cong yxy$. 
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**Problem**

Classify all finitely universal semigroup varieties.
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Classify all semigroup varieties whose subvariety lattice satisfies a non-trivial lattice identity.
“We can see only a short distance ahead, but we can see plenty there that needs to be done.” (Alan M. Turing)
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