Functorial algebras and coalgebras

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Scheme of talk

	structural problems	representations
old period		
new period		

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old period	**I.**	**II.**
new period	**III.**	**IV.**

I. Structural properties (old period)

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Alg\ F : \quad
\begin{array}{ccccc}
FQ & \xrightarrow{Fh} & FQ' & \xrightarrow{Fh'} & FQ'' \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \delta''} \\
Q & \xrightarrow[h]{} & Q' & \xrightarrow[h']{} & Q''
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[Bialgebras $A(F, G)$]

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[Bialgebras $A(F, G)$]

The category $\mathbb{A}(\Sigma)$ of all universal algebras of the type (Σ, ar) [ar being "the arity function" $\Sigma \to$Card] is just $AlgF_\Sigma$ for the polynomial functor $F_\Sigma : Set \to Set$.

$$F_\Sigma X = \coprod_{\sigma \in \Sigma} X^{ar\sigma}$$

The oldest bibliography:

- O. Wyler: Operational categories, Proc. Conference on categorical algebra Springer Verlag Berlin-Haidelberg-New York (1966), 295-316

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- V. Trnková: Some properties of set functors, Comment. Math. Univ. Carolinae 10 (1969), 323-352

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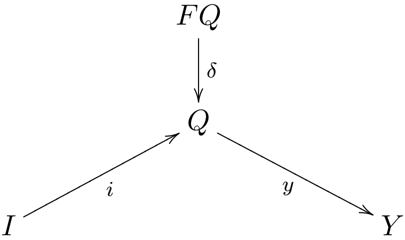

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y: output morphism

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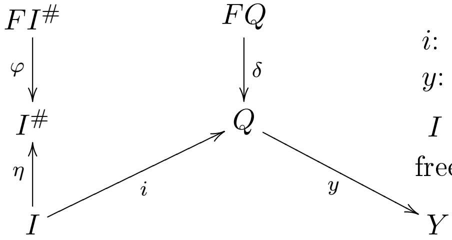

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$\left.\begin{array}{c} I \xrightarrow{\eta} I^{\#} \leftarrow FI^{\#} \\ \text{free algebra over} I \end{array}\right\}$ input process

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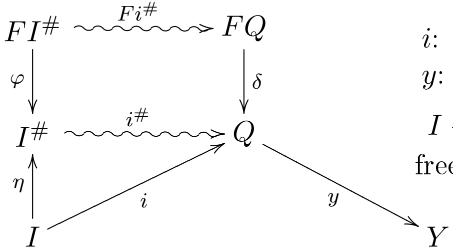

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free algebra over$I$ } input process

$b = y \circ i^{\#}$ is the behavior of the machine

minimal realization of a given behavior

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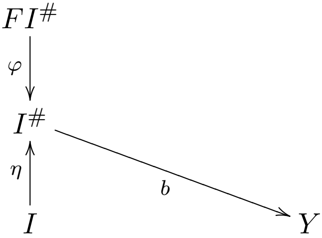

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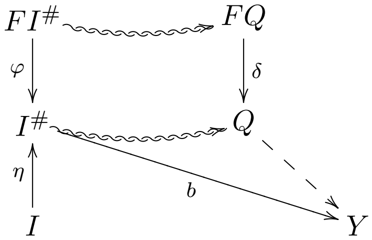

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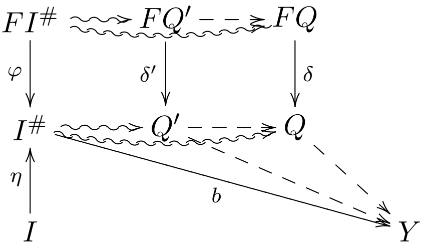

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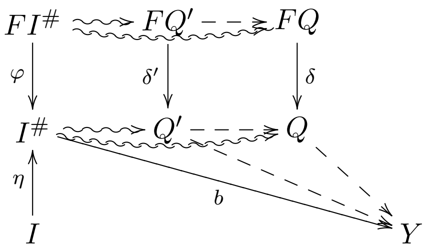
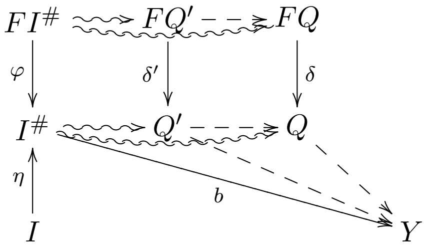


A) Which functors are input processes?

B) Which input process admits minimal realization?

minimal realization of a given behavior



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Varietor

Iterative constructions

Iterative constructions

Knaster-Tarski construction of the least fix-point of a monotone map $F : \mathcal{K} \to \mathcal{K}$, $\mathcal{K}$ complete lattice:

$$\begin{array}{ccccccccccccc} 0 & \leq & F0 & \leq & FF0 & \leq & \dots & supF^i0 & \leq & F(\dots) & \leq & \dots \\ || & & || & & || & & & || & & || & & \\ W_0 & & W_1 & & W_2 & & & W_\omega & & W_{\omega+1} & & \end{array}$$

The constructions stops if $FW_\alpha = W_\alpha$

Initial-algebra-construction: $\mathcal{K}$ category, $F : \mathcal{K} \to \mathcal{K}$ endofunctor, $\mathcal{K}$ has an initial object I (ϕ in Set, $\{0\}$ in Vect $\ldots$) $\mathcal{K}$ has colimits of chains

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$$\begin{array}{ccccccccc} I & \overset{w_{0,1}}{\rightsquigarrow} & FI & \overset{w_{1,2}}{\rightsquigarrow} & F^2I & \rightsquigarrow \dots \rightsquigarrow & colim(F^iI, w_{i,\omega}) & \rightsquigarrow & F(\dots) & \rightsquigarrow \dots \\ \| & & \| & & \| & & \| & & \| & \\ W_0 & & W_1 & & W_2 & F_{w_{i,i+1}} = w_{i+1,i+2} & W_\omega & & W_{\omega+1} & \\ & & & & & w_{i,j} \circ w_{j,s} = w_{i,s} & & & & \end{array}$$

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Then $(W_\alpha, (w_{\alpha,\alpha+1})^{-1})$ is the initial F-algebra.

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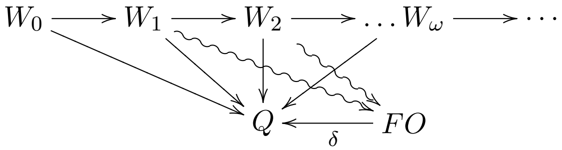

Adámek: free-algebra-construction

$$FI$$

$$\begin{array}{c} I \\ \| \\ W_0 \end{array}$$

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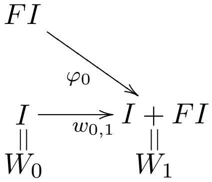

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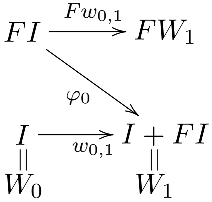

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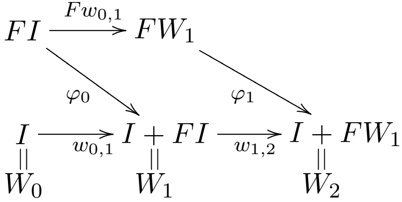

Adámek: free-algebra-construction

$$\begin{array}{ccccc} FI & \xrightarrow{Fw_{0,1}} & FW_1 & \xrightarrow{Fw_{1,2}} & FW_2 \\ & \searrow^{\varphi_0} & & \searrow^{\varphi_1} & \\ I & \xrightarrow[w_{0,1}]{} & I+FI & \xrightarrow[w_{1,2}]{} & I+FW_1 \\ \| & & \| & & \| \\ W_0 & & W_1 & & W_2 \end{array}$$

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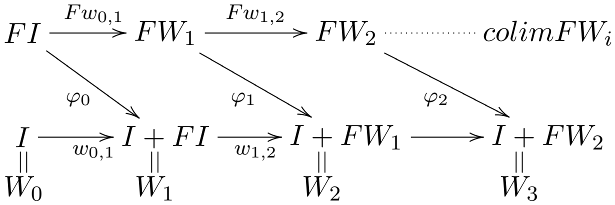

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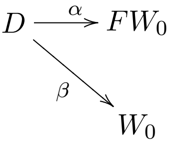
$w_{\alpha,\alpha+1} : W_\alpha \to W_{\alpha+1} = I + FW_\alpha$ is an isomorphism
Then $(W_\alpha, (w_{\alpha,\alpha+1})^{-1} \circ \varphi_{\alpha+1})$ and $\eta : I \to W_\alpha$ with $\eta = w_{0,\alpha}$ is a free F-algebra over I.

Koubek: push-out construction



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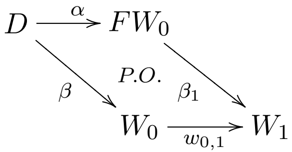

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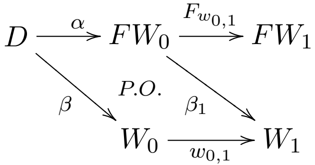

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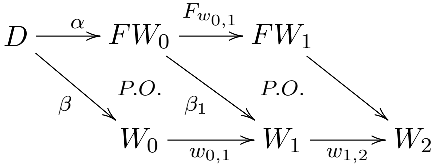

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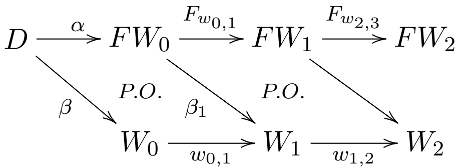

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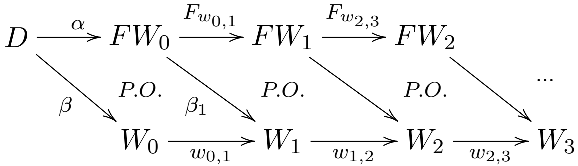

Koubek: push-out construction

$$
\begin{array}{ccccccccc}
D & \xrightarrow{\alpha} & FW_0 & \xrightarrow{F_{w_{0,1}}} & FW_1 & \xrightarrow{F_{w_{2,3}}} & FW_2 & & \\
 & \searrow^{\beta} & P.O. & \searrow^{\beta_1} & P.O. & \searrow & P.O. & \searrow & \cdots \\
 & & W_0 & \xrightarrow[w_{0,1}]{} & W_1 & \xrightarrow[w_{1,2}]{} & W_2 & \xrightarrow[w_{2,3}]{} & W_3
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If $\alpha : D \to FW_0$ is mono, then (W_0, β) is a partial F-algebra and if the push-out construction stops, we get its free completion.

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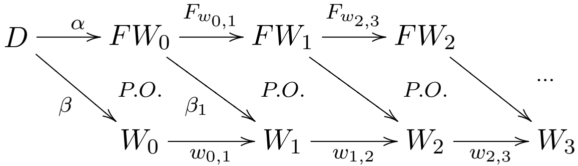

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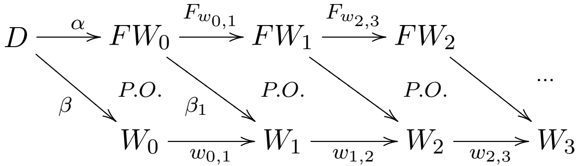

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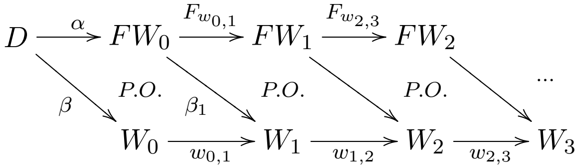

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i.e. are all initial (or free or ...) algebras constructive (in the sense that they are results of the above constructions?)
YES in *Set*
NO generally

J. Adámek, V. Trnková:

Automata and Algebras in
Categories
Kluwer Academic Publishers 1990

II. Representations (old period)

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It was an idea of J. R. Isbell to generalize it to the investigation of full embeddings of categories.
A functor $F : \mathcal{K} \to \mathcal{H}$ is a *full embedding* if it is isofunctor onto full subcategory (i.e. F maps $\mathcal{K}(a, b)$ *onto* $\mathcal{H}(Fa, Fb)$.

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A category $\mathcal{K}$ is called *algebraically universal* if every category $\mathbb{A}(\Sigma)$ of universal algebras of an arbitrary signature Σ can be fully embedded in it.

Many categories were proved to be alg-universal (Hedrlín, Pultr, Vopěnka, Kučera, Sichler, ...).

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Theorem(V. Koubek 1983): For a varietor $F : Set \to Set$, $Alg\ F$ is alg-universal iff F satisfies at least one of the following three conditions:

1. Ident+Ident is subfunctor of F
2. F is faithful and has an unattainable cardinal
3. F is not connected and has an unattainable cardinal.

[A set PX is called an *increase* of a functor $F : Set \to Set$ on a set X if

$$PX = FX \setminus \bigcup_{\substack{f:Y\to X\\|Y|<|X|}} (Ff)[FY]$$

$|X|$ is *unattainable* iff $PX \neq 0$]

Example:
Alg (2) is alg-universal; its variety given by

$$x \cdot x = y \cdot y$$

is not alg-universal; but $Alg(2{,}0)$ with

$$x \cdot x = y \cdot y$$

is alg-universal.

III. Structural problems (new period)

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J. M. Rutten: Universal coalgebra: a system theory
Theoret. Comp. Sci. 249 (2000), 3-80
H. P. Gumm and T. Schröder: Types and coalgebraic structure
Alg. Universalis 53 (2005), 229-252

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- J. Adámek and V. T.: Initial algebras and terminal coalgebras in many-sorted sets, sent for publication
- J. Adámek and V. T.: Relatively terminal coalgebras, in preparation

Selected parts:

- Let $F : \mathcal{K} \to \mathcal{K}$ admit terminal coalgebra. Must it be obtained by corresponding iterative construction?
  YES for $\mathcal{K} = Set$, many sorted sets, vector spaces

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For initial algebras the result is analogous

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Open problem: Which functors $Set \to Set$ admit terminal coalgebras?

IV. Representations

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- J. Sichler and V. T.: On universal categories of coalgebras, accepted in Alg. Universalis
- V. Koubek, J. Sichler and V. T.: Universality of categories of coalgebras, accepted in Appl. Cat. Structures

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Statements:

- if F preserves intersections, then $CoalgF$ is universal iff F is not linear (i.e. F is not of the form $FX = X \times A + B$ where A and B are constants).
- if F preserves non-void preimages then $CoalgF$ is universal iff F does not preserve finite unions of non-empty set.

Examples:
$F_1(X) = X \times X$: both $AlgF_1$ and $CoalgF_1$ are universal.
$F_2(X) = X \times X/_\Delta$: $CoalgF_2$ is universal, $AlgF_2$ not.
$F_3(X) = X + X$: $AlgF_3$ *is* universal $CoalgF_3$ is not.

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Let $\beta : Set \to Set$ be an ultrafilter functor. Then

- $Coalg\beta$ contains rigid proper class of coalgebras.
- But it is not universal, a monoid on 13 elements cannot be represented in it.