Categorically-algebraic topology

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Outline

Categorical approach to topology

- There exists a convenient approach to topological spaces:

Step 1. The backward powerset operator $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ with $\mathbf{Set}$ (resp. $\mathbf{CBAlg}$) the category of sets (resp. complete Boolean algebras) and $(X \xrightarrow{f} Y)^{\leftarrow} = \mathbf{2}^X \xrightarrow{(f^{\leftarrow})^{op}} \mathbf{2}^Y$, $f^{\leftarrow}(\alpha) = \alpha \circ f$.

Step 2. The topological theory, which is just the forgetful functor $\mathbf{CBAlg} \xrightarrow{\|-\|} \mathbf{Frm}$ to the category $\mathbf{Frm}$ of frames, describing the underlying algebraic structure of topological spaces.

Step 3. The category $\mathbf{Top}$ of topological spaces and continuous maps, whose objects are pairs (X, τ) for τ (topology) a subframe of $\|\mathbf{2}^X\|$, and whose morphisms $(X,\tau) \xrightarrow{f} (Y,\sigma)$ are maps $X \xrightarrow{f} Y$ with $(\|f^{\leftarrow}\|)^{\rightarrow}(\sigma) \subseteq \tau$ (continuity).

Poslat topology of S. E. Rodabaugh

1983: S. E. Rodabaugh considers the backward powerset theory $\mathbf{Set} \times \mathbf{CBAlg}^{op} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ defined by the formula

$$((X, L) \xrightarrow{(f,\varphi)} (Y, M))^{\leftarrow} = L^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} M^Y,$$
$$(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f,$$

and topological spaces (X, L, τ) with τ a subframe of $\|L^X\|$, resulting in variable-basis lattice-valued topology.

1991: S. E. Rodabaugh develops strict categorical foundations for his theory calling it point-set lattice-theoretic (poslat) topology.

1999: Poslat topology becomes a standard in the fuzzy community.

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1993: Motivated by a promising idea of U. Höhle, T. Kubiak and A. Šostak consider topological spaces as tuples $(X, L, M, \mathcal{T})$ with M a frame, and $L^X \xrightarrow{\mathcal{T}} M$ a map fulfilling several requirements.

2003: C. Guido suggests considering extended topological spaces $(X, L, M, \alpha, \mathcal{T})$ with $\alpha \in L^X$ and $\{\beta \in L^X \mid \beta \leqslant \alpha\} \xrightarrow{\mathcal{T}} M$.

2009: T. Kubiak and A. Šostak develop strict foundations for their theory calling it (L, M)-fuzzy topology. The theory becomes the second major approach to many-valued topological structures.

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Topological systems of S. Vickers

1989: S. Vickers proposes topological systems as a tool for doing pointless topology with. Their category has both the categories **Top** and $\mathbf{Frm}^{op}$ as full subcategories with "nice" properties.

Definition 1

A topological system is a triple (X, L, κ) with X a set, L a frame, and $L \xrightarrow{\kappa} \mathbf{2}^X$ a frame homomorphism.

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Soft sets of D. Molodtsov

1999: D. Molodtsov introduces soft sets as a promising tool to deal with uncertainty, and shows that his concept includes the notion of fuzzy set.

Definition 2

Given a set X, a soft set over X is a pair $(Y, \Vdash)$ with Y a set and $Y \xrightarrow{\Vdash} \mathbf{2}^X$ a map.

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Common framework for everything

Aim: This talk introduces a new way of approaching topological structures, which is induced by recent developments in lattice-valued topology, and is deemed to incorporate both crisp and many-valued settings.

Machinery: Based in category theory and universal algebra, the framework is called categorically-algebraic (catalg) topology, to underline its motivating theories, and to distinguish it from the poslat topology of S. E. Rodabaugh.

Advantage: The new setting includes all approaches to lattice-valued topology, as well as pointless topology of S. Vickers. It also starts a completely new area of study called soft topology, which is induced by the concept of soft set of D. Molodtsov.

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Ω-algebras and Ω-homomorphisms

Definition 3

Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers.

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-ary primitive operations on A).
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f^{n_\lambda}$ for every $\lambda \in \Lambda$.
- **Alg**(Ω) is the construct of Ω-algebras and Ω-homomorphisms, with the underlying functor denoted by $| - |$.

Varieties of algebras

Definition 4

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.

- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety are called algebras (resp. homomorphisms).
- The categorical dual of a given variety $\mathbf{A}$ is denoted by $\mathbf{LoA}$, whose objects (resp. morphisms) are called localic algebras (resp. homomorphisms).
- Given a subclass $\Omega' \subseteq \Omega$, an Ω'-reduct of $\mathbf{A}$ is a pair $(\|-\|, \mathbf{B})$ with $\mathbf{B}$ a variety of Ω'-algebras and $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ a concrete functor.

Powerset theories

Definition 5

A variety-based backward powerset theory (vbp-theory) in a given category $\mathbf{X}$ (ground category of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{LoA}$.

Lemma 6

Given a variety $\mathbf{A}$*, every subcategory* $\mathbf{C}$ *of* $\mathbf{LoA}$ *induces a functor* $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{S}=(-)^{\leftarrow}} \mathbf{LoA}$, $((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$ *with* $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

- ! $\mathbf{S}_A$ is the subcategory of $\mathbf{LoA}$ with the only morphism 1_A.
- ! $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ (fixed-basis approach, with the full setting being variable-basis approach) is denoted by $(-)_A^{\leftarrow}$.

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Examples of powerset theories

Example 7

1. $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}_{2}} \mathbf{LoCBAlg}$, where $\mathbf{2} = \{\bot, \top\}$, provides the above-mentioned backward powerset operator.
2. $\mathbf{Set} \times \mathbf{S}_{\mathbb{I}} \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_{\mathbb{I}}} \mathbf{DmLoc}$ (DeMorgan frames), where $\mathbb{I} = [0,1]$ is the unit interval, provides the fixed-basis fuzzy approach of L. A. Zadeh.
3. $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}=(-)^{\leftarrow}_{L}} \mathbf{LoUQuant}$ (unital quantales) provides the fixed-basis L-fuzzy approach of J. A. Goguen.
4. $\mathbf{Set} \times \mathbf{C} \xrightarrow{\mathcal{R}=(-)^{\leftarrow}} \mathbf{LoSQuant}$ (semi-quantales) provides the variable-basis poslat approach of S. E. Rodabaugh.

Topological theories

Definition 8

Let $\mathbf{X}$ be a category and let $\mathcal{T}_I = ((P_i, (\| - \|_i, \mathbf{B}_i)))_{i \in I}$ be a set-indexed family with $\mathbf{X} \xrightarrow{P_i} \mathbf{LoA}_i$ a vbp-theory in $\mathbf{X}$ and $(\|-\|_i, \mathbf{B}_i)$ a reduct of $\mathbf{A}_i$ for $i \in I$. A composite variety-based topological theory (cvt-theory) in $\mathbf{X}$ induced by $\mathcal{T}_I$ is the functor $\mathbf{X} \xrightarrow{T_I} \prod_{i \in I} \mathbf{LoB}_i$, defined by commutativity of the diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{P_j} & \mathbf{LoA}_j \\ {\scriptstyle T_I}\downarrow & & \downarrow{\scriptstyle \|-\|_j^{op}} \\ \prod_{i \in I} \mathbf{LoB}_i & \xrightarrow[\Gamma_j]{} & \mathbf{LoB}_j \end{array}$$

for $j \in I$, where Γ_j is the respective projection functor.

! A cvt-theory induced by a singleton family is denoted by T.

Categorically-algebraic topological spaces

Definition 9

Let T_I be a cvt-theory in a category $\mathbf{X}$. $\mathbf{CTop}(T_I)$ is the concrete category over $\mathbf{X}$, whose

objects (composite variety-based topological spaces or T_I-spaces) are pairs $(X, (\tau_i)_{i \in I})$ with X an $\mathbf{X}$-object and τ_i a subalgebra of $T_i(X)$ for $i \in I$ ($(\tau_i)_{i \in I}$ is called composite variety-based topology or T_I-topology on X), and whose

morphisms $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$, which satisfy $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ for $i \in I$ (composite variety-based continuity or T_I-continuity).

! The category $\mathbf{CTop}(T)$ is denoted by $\mathbf{Top}(T)$.

Examples of categorically-algebraic topology

Example 10

1. **Top**$((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the classical category **Top** of topological spaces and continuous maps.
2. **Top**$((\mathcal{P}, \mathbf{CSL}))$ is isomorphic to the category **Cls** of closure spaces and continuous maps of D. Aerts.
3. **CTop**$(((\mathcal{P}, \mathbf{Frm}))_{i \in \{1,2\}})$ is isomorphic to the category **BiTop** of bitopological spaces and bicontinous maps of J. C. Kelly.
4. **Top**$((\mathcal{Z}, \mathbf{Frm}))$ is isomorphic to the category $\mathbb{I}$-**Top** of fixed-basis fuzzy topological spaces of C. L. Chang.
5. **Top**$((\mathcal{G}, \mathbf{UQuant}))$ is isomorphic to the category L-**Top** of fixed-basis L-fuzzy topological spaces of J. A. Goguen.
6. **Top**$((\mathcal{R}, \mathbf{USQuant}))$ is isomorphic to the category **C**-**Top** for variable-basis poslat topology of S. E. Rodabaugh.

Lattice-valued algebras

Definition 11

Let **A**, **L** be varieties, let **CSLat**($\bigvee$) ($\bigvee$-semilattices) be a reduct of **L** and let **C** be a subcategory of **L**.

- An $(\mathbf{A},\mathbf{C})$-algebra is a triple (A,μ,L) with A an **A**-algebra, L a **C**-algebra and $|A| \xrightarrow{\mu} |L|$ a map such that for every $\lambda \in \Lambda$ and every $a_i \in A$ for $i \in n_\lambda$, $\bigwedge_{i\in n_\lambda} \mu(a_i) \leq \mu(\omega^A_\lambda(\langle a_i\rangle_{n_\lambda}))$.
- An $(\mathbf{A},\mathbf{C})$-homomorphism $(A_1,\mu_1,L_1) \xrightarrow{(\varphi,\psi)} (A_2,\mu_2,L_2)$ is an $\mathbf{A}\times\mathbf{C}$-morphism $(A_1,L_1) \xrightarrow{(\varphi,\psi)} (A_2,L_2)$ fulfilling the property $\psi\circ\mu_1(a) \leq \mu_2\circ\varphi(a)$ for every $a \in A_1$.
- **C**-**A** is the category, concrete over $\mathbf{A}\times\mathbf{C}$, comprising $(\mathbf{A},\mathbf{C})$-algebras and $(\mathbf{A},\mathbf{C})$-homomorphisms.

Lattice-valued topological theories

Definition 12

Let T_I be a cvt-theory in a category $\mathbf{X}$, let $(\mathbf{L}_i)_{i\in I}$ be a family of extensions of $\mathbf{CSLat}(\bigvee)$, and let $\mathbf{C}_i$ be a subcategory of $\mathbf{LoL}_i$ for $i \in I$. An $\mathbb{L}_I$-valued cvt-theory in $\mathbf{X}$ induced by T_I and $(\mathbf{C}_i)_{i\in I}$ is the pair $(T_I, \mathbb{L}_I)$ with $\mathbb{L}_I$ the category $\prod_{i\in I}\mathbf{C}_i$.

! The category $\mathbb{L}_I$ induced by a singleton family is denoted by $\mathbb{L}$.

Remark

The setting of Definition 12 allows not just different underlying lattices for fuzzification (variable-basis framework), but actually different varieties for these lattices to come from.

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Lattice-valued catalg spaces

Definition 13

Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in a category $\mathbf{X}$. $\mathbb{L}_I\mathbf{CTop}(T_I)$ is the concrete category over $\mathbf{X} \times \mathbb{L}_I$, whose

objects ($\mathbb{L}_I$-valued T_I-spaces) are triples $(X, (\mathcal{T}_i)_{i\in I}, (L_i)_{i\in I})$ with X in $\mathbf{X}$, $(L_i)_{i\in I}$ in $\mathbb{L}_I$ and $T_i(X) \xrightarrow{\mathcal{T}_i} L_i$ a $(\mathbf{B}_i, \mathbf{LoC}_i)$-algebra for $i \in I$ ($(\mathcal{T}_i)_{i\in I}$ is called $\mathbb{L}_I$-valued T_I-topology on X), and whose

morphisms $(X, (\mathcal{T}_i)_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (\mathcal{S}_i)_{i\in I}, (M_i)_{i\in I})$ are $\mathbf{X} \times \mathbb{L}_I$-morphisms $(X, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (M_i)_{i\in I})$, for which $(T_i(X), \mathcal{T}_i, L_i) \xrightarrow{(T_i f, \psi_i)} (T_i(Y), \mathcal{S}_i, M_i)$ is a $\mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i)$-morphism for $i \in I$ ($\mathbb{L}_I$-valued T_I-continuity).

The underlying functor to the ground category is denoted by $|-|$.

! The category $\mathbb{L}\mathbf{CTop}(T)$ is denoted by $\mathbb{L}\mathbf{Top}(T)$.

Examples of lattice-valued catalg topology

Example 14

1. $\mathbb{L}\mathbf{Top}((\mathcal{S}^{\mathbf{S}_L}_{\mathbf{CLat}}, \mathbf{Frm}, \mathbf{S}^{\mathbf{CDCLat}}_M))$, with **CLat** being the variety of complete lattices and **CDCLat** its subcategory of completely distributive lattices, is the theory of (L,M)-fuzzy topological spaces of T. Kubiak and A. Šostak.
2. $\mathbb{L}\mathbf{Top}((\mathcal{P}, \mathbf{Frm}, \mathbf{S}^{\mathbf{DMLoc}}_M))$ provides the approach of U. Höhle.
3. $\mathbb{L}_I\mathbf{CTop}((T_I, \mathbb{L}_I))$ with $\mathbf{C}_i = \mathbf{S}_2^{\mathbf{CSLat}(\bigvee)}$ for $i \in I$, is isomorphic to the category $\mathbf{CTop}(T_I)$.

Main result

Theorem 15

The concrete category $(\mathbb{L}_I\mathbf{CTop}(T_I), |-|)$ is topological over its ground category $\mathbf{X} \times \mathbb{L}_I$.

- Meta-mathematically restated, one is doing topology when working in the category $\mathbb{L}_I\mathbf{CTop}(T_I)$.
- Given a topological structure, one can find the variety with the minimum requirements on its algebras, to preserve the "main" properties of the structure (characterizing variety). The corresponding category of lattice-valued catalg spaces (characterizing category) is then topological.

Lattice-valued catalg topological systems ...

Definition 16

Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in $\mathbf{X}$. $\mathbb{L}_I\mathbf{CTopSys}(T_I)$ is the concrete category over $\mathbf{X} \times (\prod_{i \in I} \mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i))$, whose

objects ($\mathbb{L}_I$-valued composite variety-based topological systems or $\mathbb{L}_I$-valued T_I-systems) are triples $(X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I})$ with X in $\mathbf{X}$, $((A_i, \mu_i, L_i))_{i \in I}$ in $\prod_{i \in I} \mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i)$ and $T_i(X) \xrightarrow{\kappa_i} B_i$ a $\mathbf{LoB}_i$-morphism for $i \in I$ ($(\kappa_i)_{i \in I}$ is called $\mathbb{L}_I$-valued composite variety-based satisfaction relation or $\mathbb{L}_I$-valued T_I-satisfaction relation on $(X, ((A_i, \mu_i, L_i))_{i \in I})$), and whose

...and their morphisms

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$$(X,(\kappa_i)_{i\in I},((A_i,\mu_i,L_i))_{i\in I}) \xrightarrow{(f,((\varphi_i,\psi_i))_{i\in I})} (Y,(\iota_i)_{i\in I},((B_i,\nu_i,M_i))_{i\in I})$$

are $\mathbf{X} \times (\prod_{i\in I} \mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i))$-morphisms

$$(X,((A_i,\mu_i,L_i))_{i\in I}) \xrightarrow{(f,((\varphi_i,\psi_i))_{i\in I})} (Y,((B_i,\nu_i,M_i))_{i\in I}),$$

which make the diagram

$$\begin{array}{ccc} T_i(X) & \xrightarrow{T_i f} & T_i(Y) \\ {\scriptstyle \kappa_i}\downarrow & & \downarrow{\scriptstyle \iota_i} \\ A_i & \xrightarrow[\varphi_i]{} & B_i \end{array}$$

commute for $i \in I$ ($\mathbb{L}_I$-valued composite variety-based continuity or $\mathbb{L}_I$-valued T_I-continuity).

Examples of lattice-valued catalg topological systems

! The category $\mathbb{L}$**CTopSys**(T) is denoted by $\mathbb{L}$**TopSys**(T).

Example 17

1. $\mathbb{L}\mathbf{TopSys}((\mathcal{P}, \mathbf{Frm}, \mathbf{S}_{\mathbf{2}}^{\mathbf{CSLat}(\bigvee)}))$ is isomorphic to the category **TopSys** of classical topological systems of S. Vickers.
2. $\mathbb{L}\mathbf{TopSys}((\mathcal{S}_{\mathbf{Frm}}^{\mathbf{Frm}}, \mathbf{Frm}, \mathbf{S}_{\mathbf{2}}^{\mathbf{CSLat}(\bigvee)}))$ is isomorphic to the category **Loc-TopSys** of lattice-valued topological systems introduced by J. T. Denniston, A. Melton and S. E. Rodabaugh.
3. $\mathbb{L}\mathbf{TopSys}((\mathcal{S}_{\mathbf{Set}}^{\mathbf{S}_K}, \mathbf{Set}, \mathbf{S}_{\mathbf{2}}^{\mathbf{CSLat}(\bigvee)}))$ is isomorphic to the category $\mathrm{Chu}(\mathbf{Set}, K)$ of Chu spaces over a set K of V. Pratt.
4. $\mathrm{Chu}(\mathbf{Set},\mathbf{2})$ is the category **IntSys** of interchange systems of J. T. Denniston, A. Melton and S. E. Rodabaugh. Interchange systems are called contexts in Formal Concept Analysis.

Main result

Theorem 18

There exists a full embedding $\mathbb{L}_I\mathbf{CTop}(T_I) \overset{G_I}{\hookrightarrow} \mathbb{L}_I\mathbf{CTopSys}(T_I)$. *If the underlying lattices of* $\mathbb{L}_I$ *are completely distributive, then*

1. *there exists a functor* $\mathbb{L}_I\mathbf{CTopSys}(T_I) \xrightarrow{\mathrm{Spat}_I} \mathbb{L}_I\mathbf{CTop}(T_I)$;
2. Spat_I *is a right-adjoint-left-inverse to* G_I;
3. $\mathbb{L}_I\mathbf{CTop}(T_I)$ *is isomorphic to a full coreflective subcategory of* $\mathbb{L}_I\mathbf{CTopSys}(T_I)$.

- Theorem 18 provides a lattice-valued catalg analogue for the spatialization procedure of S. Vickers, restoring a significant part of the classical framework.

Soft algebras

Definition 19

Let $\mathbf{A}$ be a variety, let A be an $\mathbf{A}$-algebra and let X be a set. A soft ($\mathbf{A}$-)algebra over A is a pair $(\Vdash, X)$, where $X \xrightarrow{\Vdash} \mathbf{2}^A$ is a map such that $\Vdash(x)$ is a subalgebra of A for every $x \in X$.

Example 20

The varieties of groups, rings, semirings, as well as quasi-varieties (in the obvious sense) of BCK/BCI-algebras provide the respective soft notions from the literature.

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Example 20

The varieties of groups, rings, semirings, as well as quasi-varieties (in the obvious sense) of BCK/BCI-algebras provide the respective soft notions from the literature.

Soft topology ...

- The next definition is induced by the concept of soft algebra.

Definition 21

Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in $\mathbf{X}$. $\mathbb{L}_I\mathbf{CSoftTop}(T_I)$ is the concrete category over $\mathbb{L}_I\mathbf{CTop}(T_I) \times (\prod_{i\in I} \mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i))$, whose objects (soft $\mathbb{L}_I$-valued T_I-spaces) are triples

$$((X, (\mathcal{T}_i)_{i\in I}, (L_i)_{i\in I}), ((\kappa_i, \varphi_i))_{i\in I}, ((B_i, \nu_i, M_i))_{i\in I})$$

such that

- $(X, (\mathcal{T}_i)_{i\in I}, (L_i)_{i\in I})$ is an $\mathbb{L}_I$-valued T_I-space;
- $((B_i, \nu_i, M_i))_{i\in I}$ is in $\prod_{i\in I} \mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i)$;
- $(T_i(X), \mathcal{T}_i, L_i) \xrightarrow{(\kappa_i, \varphi_i)} (B_i, \nu_i, M_i)$ is in $\mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i)$ for $i \in I$;

$(((\kappa_i, \varphi_i))_{i\in I}$ is called soft $\mathbb{L}_I$-valued T_I-topology on $((X, (\mathcal{T}_i)_{i\in I}, (L_i)_{i\in I}), ((B_i, \nu_i, M_i))_{i\in I}))$, and whose

. . . and soft continuity

morphisms

$$((X,(\mathcal{T}_i)_{i\in I},(L_i)_{i\in I}),((\kappa_i,\varphi_i))_{i\in I},((B_i,\nu_i,M_i))_{i\in I})\xrightarrow{((f,(\phi_i)_{i\in I}),((\xi_i,o_i))_{i\in I})} ((Y,(\mathcal{S}_i)_{i\in I},(N_i)_{i\in I}),((\iota_i,\psi_i))_{i\in I},((C_i,\sigma_i,O_i))_{i\in I})$$

are $\mathbb{L}_I\mathbf{CTop}(T_I)\times(\prod_{i\in I}\mathbf{Lo}(\mathbf{LoC}_i\text{-}\mathbf{B}_i))$-morphisms

$$((X,(\mathcal{T}_i)_{i\in I},(L_i)_{i\in I}),((B_i,\nu_i,M_i))_{i\in I})\xrightarrow{((f,(\phi_i)_{i\in I}),((\xi_i,o_i))_{i\in I})} ((Y,(\mathcal{S}_i)_{i\in I},(N_i)_{i\in I}),((C_i,\sigma_i,O_i))_{i\in I}),$$

which make the diagram

$$\begin{array}{ccc} T_i(X) & \xrightarrow{T_i f} & T_i(Y) \\ {\scriptstyle\kappa_i}\downarrow & & \downarrow{\scriptstyle\iota_i} \\ B_i & \xrightarrow[\xi_i]{} & C_i \end{array}$$

commute for $i\in I$ (soft $\mathbb{L}_I$-valued T_I-continuity).

Example of soft lattice-valued catalg topology

Example 22

The (non-full) subcategory **S** of the category $\mathbb{L}_I\mathbf{CSoftTop}(T_I)$, which comprises all objects

$$((X,(\mathcal{T}_i)_{i\in I},(L_i)_{i\in I}),((1_{T_i(X)},1_{L_i}))_{i\in I},((T_i(X),\mathcal{T}_i,L_i))_{i\in I}),$$

together with all morphisms

$$\begin{array}{c}((X,(\mathcal{T}_i)_{i\in I},(L_i)_{i\in I}),((1_{T_i(X)},1_{L_i}))_{i\in I},((T_i(X),\mathcal{T}_i,L_i))_{i\in I})\\ \Big\downarrow{\scriptstyle ((f,(\phi_i)_{i\in I}),((T_i f,\phi_i))_{i\in I})}\\ ((Y,(\mathcal{S}_i)_{i\in I},(M_i)_{i\in I}),((1_{T_i(Y)},1_{M_i}))_{i\in I},((T_i(Y),\mathcal{S}_i,M_i))_{i\in I}),\end{array}$$

is isomorphic to the category $\mathbb{L}_I\mathbf{CTop}(T_I)$.

Contribution of the talk

- The talk introduced a new approach to topological structures called (lattice-valued) categorically-algebraic topology.
- The framework incorporates crisp and many-valued topology (erasing the border between them in some cases), pointless topology and soft topology.
- It appears that the currently dominating many-valued theory of S. E. Rodabaugh does not deviate significantly from the machinery of the crisp approach, whereas the framework of T. Kubaik and A. Šostak gives a truly lattice-valued setting.

Soft algebra homomorphisms

Definition 23

Let **A** ba a variety, let $(X_1,\Vdash_1,A_1)$, $(X_2,\Vdash_2,A_2)$ be soft **A**-algebras. A soft (**A**-)algebra homomorphism $(X_1,\Vdash_1,A_1) \xrightarrow{(f,\varphi)} (X_2,\Vdash_2,A_2)$ is a **Set** $\times$ **A**-morphism $(X_1,A_1) \xrightarrow{(f,\varphi)} (X_2,A_2)$, which satisfies the following diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad f \quad} & X_2 \\ {\scriptstyle \Vdash_1}\downarrow & \leq & \downarrow{\scriptstyle \Vdash_2} \\ \mathbf{2}^{A_1} & \xrightarrow[\varphi^{\rightarrow}]{} & \mathbf{2}^{A_2}. \end{array}$$

The category **SoftA** comprises soft **A**-algebras and soft **A**-algebra homomorphisms, and is concrete over **Set** $\times$ **A**.

Problem: Develop the theory of soft algebras through investigation of the properties of the category **SoftA**.

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Thank you for your attention!