Higman embeddings

Mark Sapir

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Groups turning into machines

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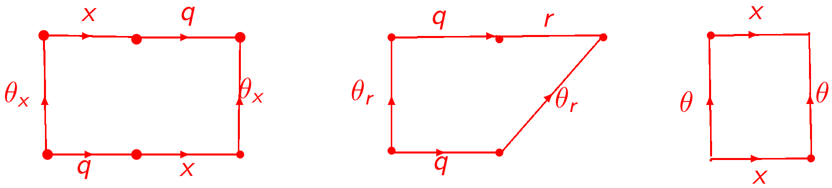

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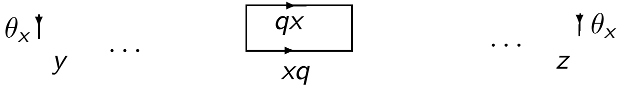

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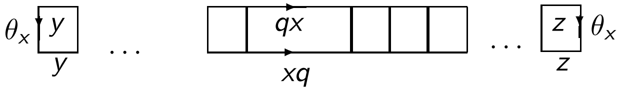

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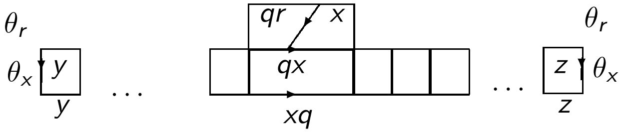

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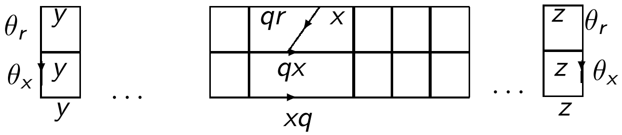

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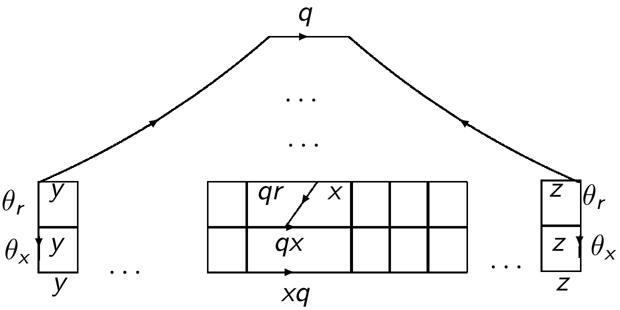

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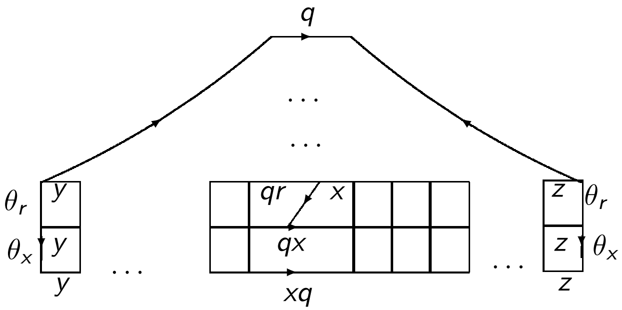

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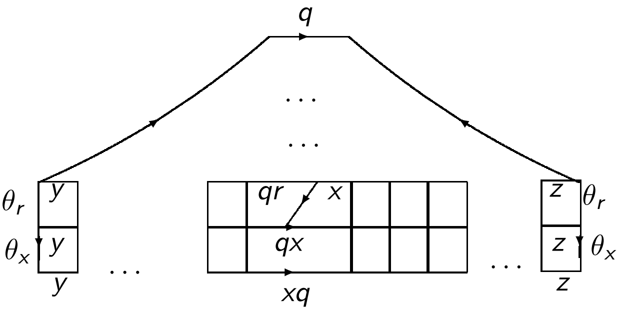

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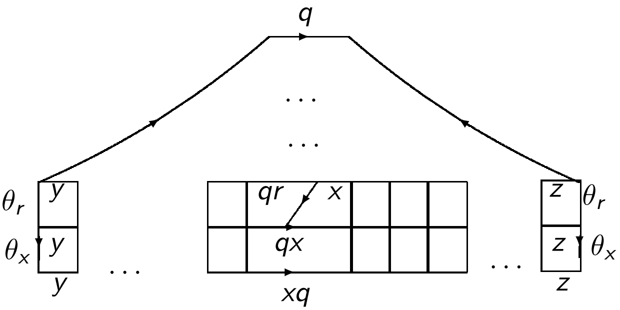

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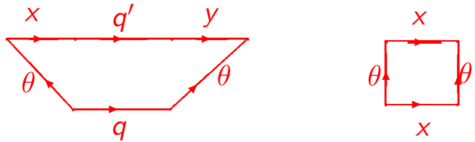

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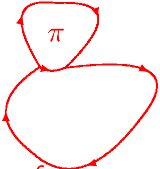

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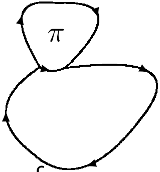

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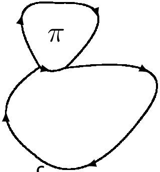

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Definition. A group is *hyperbolic* if its Dehn function is linear.

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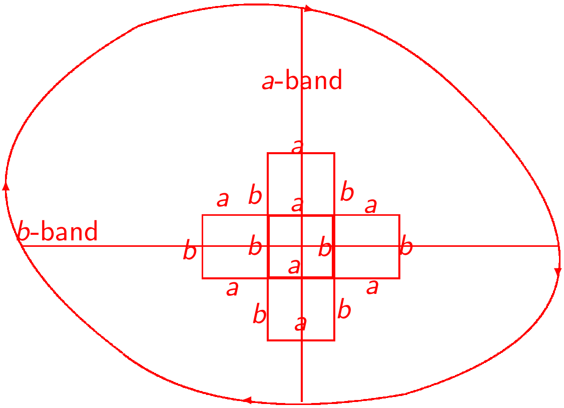

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for every word u in the alphabet $\{a_1, a_2\}$, $N = 28$, $k_1, ..., k_N, q_1, q_2, q_3$ are new letters, and the words between consecutive $k's$ are copies of $uq_1uq_2uq_3$ written in disjoint alphabets.

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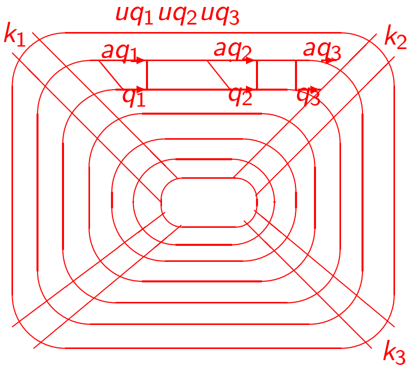

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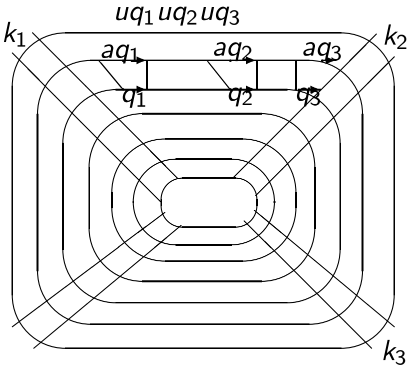

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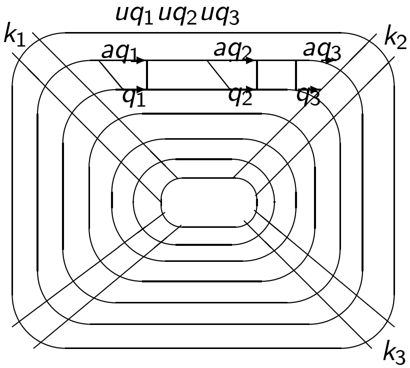

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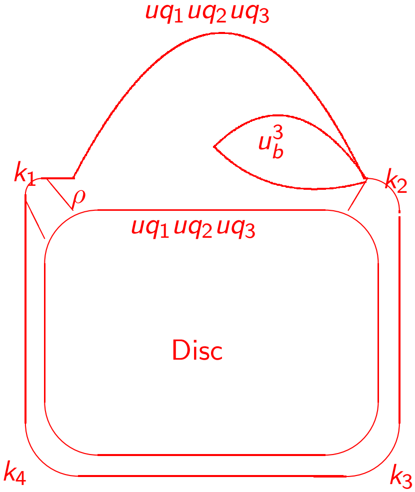

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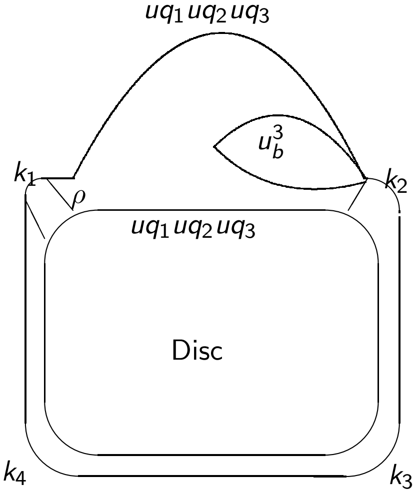

Here $a_i\rho = \rho a_i b_i$ $i = 1, 2$, plus commutativity relations, so $\rho K(u) = K(u)u_b^n\rho$. Hence $u_b^n = 1$

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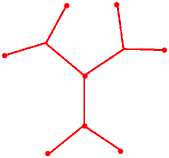

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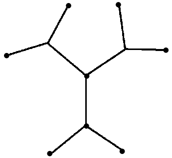

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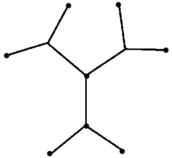

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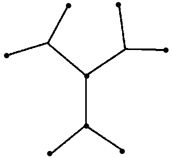

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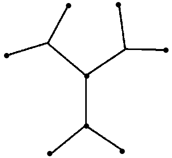

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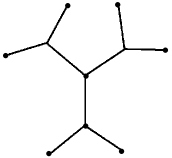

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