DYADIC POLYGONS

A. B. ROMANOWSKA
Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-661 Warsaw, Poland
J. D. H. SMITH
Department of Mathematics, Iowa State University, Ames, Iowa, 50011, USA
K. MATCZAK,
Faculty of Civil Engineering Mechanics and Petrochemistry in Płock, Warsaw University of Technology, 09 400 Płock, Poland

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REAL AFFINE SPACES

Given a vector space (a module) A over a field (a subring R of) $\mathbb{R}$.

An **affine space** A **over** R (or **affine** R**-space**) is the algebra

$$\Big(A, \sum_{i=1}^{n} x_i r_i \Big| \sum_{i=1}^{n} r_i = 1\Big).$$

In the case $2 \in R$ is invertible, this algebra is equivalent to

$$(A, \underline{R}),$$

where

$$\underline{R} = \{\underline{f} \mid f \in R\}$$

and

$$xy\underline{f} = \underline{f}(x,y) = x(1-f) + yf.$$

REAL CONVEX SETS and BARYCENTRIC ALGEBRAS

Let R be a subfield of $\mathbb{R}$ and $I^o :=]0,1[= (0,1) \subset R$.

Convex subsets of affine R-spaces are I^o-subreducts $(A, \underline{I}^o)$ of R-spaces.

Real **polytopes** are finitely generated convex sets, real **polygons** are finitely generated convex subsets of R^2.

The class C of convex sets generates the variety BA of **barycentric algebras**.

DYADIC CONVEX SETS

Consider the ring

$$\mathbb{D} = \mathbb{Z}[1/2] = \{m2^{-n} \mid m, n \in \mathbb{Z}\}$$

of dyadic rational numbers.

A *dyadic convex set* is the intersection of a real convex set with the space $\mathbb{D}^k$.

A *dyadic polytope* is the intersection of a real polytope and $\mathbb{D}^k$, with vertices in $\mathbb{D}^k$.

A *dyadic triangle* and *dyadic polygon* are (respectively) the intersection with $\mathbb{D}^2$ of a triangle or polygon in $\mathbb{R}^2$, with vertices in $\mathbb{D}^2$.

Dyadic intervals form the one-dimensional analogue.

REAL VERSUS DYADIC

• Real polytopes are barycentric algebras $(A, \underline{I}^o)$.

Dyadic polytopes are algebras $(A, \underline{\mathbb{D}}_1^o)$,
where $\mathbb{D}_1^o =]0,1[\cap \mathbb{D}$.

Proposition Each dyadic polytope $(A, \underline{\mathbb{D}}_1^o)$ is equivalent to $(A, \cdot) = (A, \frac{1}{2}(x+y))$.

Note that the operation $\cdot$ is

idempotent: $x \cdot x = x$,
commutative: $x \cdot y = y \cdot x$,
entropic (medial): $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$.

Hence:
dyadic polytopes are *commutative binary modes* (or CB-modes).

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent**:

$$x...x\omega = x,$$

  for each n-ary $\omega \in \Omega$, and

- **entropic**:

$$(x_{11}...x_{1n}\omega)...(x_{m1}...x_{mn}\omega)\,\varphi$$

$$= (x_{11}...x_{m1}\varphi)...(x_{1n}...x_{mn}\varphi)\,\omega.$$

  for all $\omega, \varphi \in \Omega$.

Affine R-spaces and barycentric algebras are modes.

REAL VERSUS DYADIC, cont.

• All real intervals are isomorphic (to the interval $I = S_1$). Each is generated by its ends. All real triangles are isomorphic (to the simplex S_2). Each is generated by its vertices.

NOT TRUE for dyadic intervals and dyadic triangles.

Example The dyadic interval $[0, 3]$ is generated by no less than 3 elements. The minimal set of generators is given e.g. by the numbers $0, 2, 3$.

• The class of convex subsets of affine $\mathbb{R}$-spaces is characterized as the subquasivariety of cancellative barycentric algebras.

NOT TRUE for the class of convex dyadic subsets of affine $\mathbb{D}$-spaces.
(K. Matczak, A. Romanowska)

PROBLEMS

Which characteristic properties of real polytopes (in particular polygons) carry over to dyadic polytopes (polygons)?

Note that dyadic polygons are described using dyadic intervals and dyadic triangles.

Problem:
Are all dyadic intervals finitely generated?
Are all dyadic triangles finitely generated?

Problem: Classify all dyadic intervals and all dyadic triangles up to isomorphism.

Isomorphisms of dyadic polytopes are described as restrictions of automorphisms of the affine dyadic spaces, members of the affine group $GA(n, \mathbb{D})$.

DYADIC INTERVALS

The isomorphism classes of dyadic intervals are determined by the orbits of $GL(1, \mathbb{D})$ on the set of nonzero dyadic numbers.

THEOREM Each interval of $\mathbb{D}$ is isomorphic to some interval $[0, k]$ (is *of type* k), where k is an odd positive integer. Two such intervals are isomorphic precisely when their right hand ends are equal.

The interval $[0, 1]$ is generated by its ends. For each positive integer k, and each integer r, the intervals $[0, k]$ and $[0, k2^r]$ are isomorphic.

THEOREM Each dyadic interval of type $k > 1$ is minimally generated by three elements, e.g. by $0, 2^n, k$, where $n = \lfloor \log_2 k \rfloor$.

DYADIC TRIANGLES AND THEIR BOUNDARY TYPES

The types m, n, k of sides of a triangle determine its *boundary type* (m, n, k).

Proposition The triangles of *right type* (i.e.with shorter side parallel to the coordinate axes) are determined uniquely up to isomorphism by its boundary type.

The boundary type does not determine a general dyadic triangle.

Proposition There are infinitely many pairwise non-isomorphic triangles of boundary type $(1, 1, 1)$.

There are triangles in $\mathbb{D}$ not isomorphic to right triangles

TYPES OF DYADIC TRIANGLES

Automorphisms of the dyadic plane $\mathbb{D}^2$ are described as elements of the affine group $GA(2, \mathbb{D})$. These automorphisms transform any of the triangles in the plane $\mathbb{D}^2$ into an isomorphic triangle.

A point $A = (p2^q, u2^v)$ of $\mathbb{D}^2$, where p, u, q and v are integers, with p and u being odd, is said to be *axial* if

$$\text{gcd}\{p, u\} \in \{p, u, 1\}\,.$$

Lemma A $\mathbb{D}$-module automorphism of the plane $\mathbb{D}^2$ transforms A into a point on one of the axes if and only if A is axial.

A classification of dyadic triangles depends on the existence of axial vertices.

Lemma Each dyadic triangle is isomorphic to a triangle ABC contained in the first quadrant, with A located at the origin. Moreover, the vertices B and C may be chosen so that they have integral coordinates.

THEOREM Each dyadic triangle, located as in the lemma, belongs to one of three **basic types** (with m, n, k positive integers):
• triangles isomorphic to right triangles $T_{m,n}$ (with vertices $(0, 0), (m, 0)$ and $(0, n)$);
• triangles isomorphic to triangles $T_{m,n,k}$ (with vertices $(-k, 0), (m, 0), (0, n)$);
• triangles in which neither B nor C is axial.

It is then shown that certain quadruples of positive integers form complete invariants for classifying dyadic triangles up to isomorphism.

FINITE GENERATION

The classification of dyadic triangles given above provided a basis for proving the following.

THEOREM Each dyadic triangle is finitely generated.

COROLLARY Each dyadic polygon is finitely generated.