Affine complete G-sets

András Pongrácz, Gábor Horváth, Peter Mayr

ELTE Budapest, Hungary

Prague, June 2010.

Grätzer

Describe the affine complete algebras.

Definition

Affine completeness

An algebra $\mathcal{A}$ is affine complete if every compatible function is a polynomial on $\mathcal{A}$.

Definition

Affine completeness

An algebra $\mathcal{A}$ is affine complete if every compatible function is a polynomial on $\mathcal{A}$.

Definition

$f : A^k \to A$ compatible : for all $\theta \in Con(\mathcal{A})$ we have $(\mathbf{a}, \mathbf{b}) \in \theta \Rightarrow (f(\mathbf{a}), f(\mathbf{b})) \in \theta$.

Definition

Affine completeness

An algebra $\mathcal{A}$ is affine complete if every compatible function is a polynomial on $\mathcal{A}$.

Definition

$f : A^k \to A$ compatible : for all $\theta \in Con(\mathcal{A})$ we have $(\mathbf{a}, \mathbf{b}) \in \theta \Rightarrow (f(\mathbf{a}), f(\mathbf{b})) \in \theta$.

- constants, fundamental operations

Definition

Affine completeness

An algebra $\mathcal{A}$ is affine complete if every compatible function is a polynomial on $\mathcal{A}$.

Definition

$f : A^k \to A$ compatible : for all $\theta \in Con(\mathcal{A})$ we have $(\mathbf{a}, \mathbf{b}) \in \theta \Rightarrow (f(\mathbf{a}), f(\mathbf{b})) \in \theta$.

- constants, fundamental operations
- polynomials

Definition

Affine completeness

An algebra $\mathcal{A}$ is affine complete if every compatible function is a polynomial on $\mathcal{A}$.

Definition

$f : A^k \to A$ compatible : for all $\theta \in Con(\mathcal{A})$ we have $(\mathbf{a}, \mathbf{b}) \in \theta \Rightarrow (f(\mathbf{a}), f(\mathbf{b})) \in \theta$.

- constants, fundamental operations
- polynomials

k-affine completeness

An algebra $\mathcal{A}$ is k-affine coplete if every compatible function of arity at most k is a polynomial on $\mathcal{A}$.

Examples

Fields, Boolean algebras

Finite fields, 2-element Boolean algebra: every function is a polynomial. These are affine complete.

Examples

Fields, Boolean algebras

Finite fields, 2-element Boolean algebra: every function is a polynomial. These are affine complete.

Z_p

Examples

Fields, Boolean algebras

Finite fields, 2-element Boolean algebra: every function is a polynomial.
These are affine complete.

Z_p

- every function is compatible

Examples

Fields, Boolean algebras

Finite fields, 2-element Boolean algebra: every function is a polynomial. These are affine complete.

Z_p

- every function is compatible
- polynomials: $\sum_{i=1}^{n} a_i x_i + b$

Examples

Fields, Boolean algebras

Finite fields, 2-element Boolean algebra: every function is a polynomial. These are affine complete.

Z_p

- every function is compatible
- polynomials: $\sum_{i=1}^{n} a_i x_i + b$
- $\Rightarrow$ not affine complete

Grätzer

Describe the affine complete algebras.

Grätzer

Describe the affine complete algebras.

First significant results:

Grätzer

Describe the affine complete algebras.

First significant results:

- Boolean algebras

Grätzer

Describe the affine complete algebras.

First significant results:

- Boolean algebras
- Bounded distributive lattices: not containing proper Boolean intervals

Some known results

- Abelian groups: K. Kaarli
- Semilattices: K. Kaarli, L. Márki, E. T. Schmidt
- Vector spaces: H. Werner
- Distributive lattices: M. Ploščica
- Stone algebras: M. Haviar, M. Ploščica
- Kleene algebras: M. Haviar, K. Kaarli, M. Ploščica

Definition

G-set

Definition

G-set

- (Ω, G) pair

Definition

G-set

- (Ω, G) pair
- underlying set: Ω

Definition

G-set

- (Ω, G) pair
- underlying set: Ω
- type: for all $g \in G$ a unary operation is given:

$$m_g : \alpha \mapsto g\alpha$$

Definition

G-set

- (Ω, G) pair
- underlying set: Ω
- type: for all $g \in G$ a unary operation is given:

$$m_g : \alpha \mapsto g\alpha$$

Transitive group actions

Definition

G-set

- (Ω, G) pair
- underlying set: Ω
- type: for all $g \in G$ a unary operation is given:

$$m_g : \alpha \mapsto g\alpha$$

Transitive group actions

- $S \leq G$, Ω: cosets of S

Definition

G-set

- (Ω, G) pair
- underlying set: Ω
- type: for all $g \in G$ a unary operation is given:

$$m_g : \alpha \mapsto g\alpha$$

Transitive group actions

- $S \leq G$, Ω: cosets of S
- $m_g(hS) = ghS$

Transitive group actions

Congruences

Transitive group actions

Congruences

- $S \leq H \leq G$

Transitive group actions

Congruences

- $S \leq H \leq G$

Transitive group actions

Congruences

- $S \leq H \leq G$

$$\rho_H = \{(aS, bS) \in \Omega \times \Omega | a, b \in G, aH = bH\}$$

Transitive group actions

Congruences

- $S \leq H \leq G$

$$\rho_H = \{(aS, bS) \in \Omega \times \Omega | a, b \in G, aH = bH\}$$

- Congruences $\leftrightsquigarrow \{H | S \leq H \leq G\}$

Transitive group actions

Congruences

- $S \leq H \leq G$

$$\rho_H = \{(aS, bS) \in \Omega \times \Omega | a, b \in G, aH = bH\}$$

- Congruences $\leftrightsquigarrow \{H | S \leq H \leq G\}$
- $Con((\Omega, G)) = [S, G] \leq L(G)$

Transitive group actions

Congruences

- $S \leq H \leq G$

$$\rho_H = \{(aS, bS) \in \Omega \times \Omega | a, b \in G, aH = bH\}$$

- Congruences $\leftrightsquigarrow \{H | S \leq H \leq G\}$
- $Con((\Omega, G)) = [S, G] \leq L(G)$

Regular G-sets

$Con(R(G)) = \{\rho_H | H \leq G\} \cong L(G)$ (subgroup lattice)

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G
f compatible on $R(G)$: for all $H \leq G$ and $a, b \in G$

$$aH = bH \Rightarrow f(a)H = f(b)H$$

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G
f compatible on $R(G)$: for all $H \leq G$ and $a, b \in G$

$$aH = bH \Rightarrow f(a)H = f(b)H$$

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G
f compatible on $R(G)$: for all $H \leq G$ and $a, b \in G$

$$aH = bH \Rightarrow f(a)H = f(b)H$$

- constants $x \mapsto g$, $g \in G$

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G
f compatible on $R(G)$: for all $H \leq G$ and $a, b \in G$

$$aH = bH \Rightarrow f(a)H = f(b)H$$

- constants $x \mapsto g$, $g \in G$
- left translations $x \mapsto gx$, $g \in G$

Regular actions

Unary compatible functions

$R(G)$: regular permutation representation of G
f compatible on $R(G)$: for all $H \leq G$ and $a, b \in G$

$$aH = bH \Rightarrow f(a)H = f(b)H$$

- constants $x \mapsto g$, $g \in G$
- left translations $x \mapsto gx$, $g \in G$
- These are the only unary polynomial functions on $R(G)$.

1-affine completeness

Theorem

Pálfy (1984.): Classification of minimal algebras.

1-affine completeness

Theorem

Pálfy (1984.): Classification of minimal algebras.

Corollary

(Ω, G) 1-affine complete $\Rightarrow$ affine complete, except:

1-affine completeness

Theorem

Pálfy (1984.): Classification of minimal algebras.

Corollary

(Ω, G) 1-affine complete $\Rightarrow$ affine complete, except:

- $|\Omega| = 2$

1-affine completeness

Theorem

Pálfy (1984.): Classification of minimal algebras.

Corollary

(Ω, G) 1-affine complete $\Rightarrow$ affine complete, except:

- $|\Omega| = 2$
- There exists a division ring D and a vector space ${}_D V$ such that $\Omega =_D V$ and $G = \{x \mapsto dx + v | d \in D, v \in V\}$

Notation

t-completeness

A group G is t-complete if $R(G)$ is a 1-affine complete G-set.

Necessary conditions

Proposition

Let G be an abelian group. Then G is not t-complete, except if G is an elementary abelian 2-group.

Necessary conditions

Proposition

Let G be an abelian group. Then G is not t-complete, except if G is an elementary abelian 2-group.

Proposition

$x \mapsto x^{|G:Z(G)|}$ (transfer) is compatible
G is t-complete $\Rightarrow$ $|G : Z(G)|$ is divisible by $exp(G)$.

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1$

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1$

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.
$\pi : (a, b) \mapsto (a, 1)$ is compatible

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.
$\pi : (a, b) \mapsto (a, 1)$ is compatible
$(a_1, b_1)C \times D = (a_2, b_2)C \times D$

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.
$\pi : (a, b) \mapsto (a, 1)$ is compatible
$(a_1, b_1)C \times D = (a_2, b_2)C \times D$
$\Rightarrow a_1 C = a_2 C$

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.
$\pi : (a, b) \mapsto (a, 1)$ is compatible
$(a_1, b_1)C \times D = (a_2, b_2)C \times D$
$\Rightarrow a_1 C = a_2 C$
$\Rightarrow (a_1, 1)C \times D = (a_2, 1)C \times D$

Necessary conditions

Lemma (Typical counterexample)

$G = A \times B$, A, B are proper subgroups and $gcd(|A|, |B|) = 1$. Then G is not t-complete.

Proof

$gcd(|A|, |B|) = 1 \Rightarrow$ subgroups of G are of the form $C \times D$.
$\pi : (a, b) \mapsto (a, 1)$ is compatible
$(a_1, b_1)C \times D = (a_2, b_2)C \times D$
$\Rightarrow a_1 C = a_2 C$
$\Rightarrow (a_1, 1)C \times D = (a_2, 1)C \times D$
π is not a polynomial function

Necessary conditions

Theorem

Let $A, B \leq G$ such that $\langle A, B \rangle = G$. If A and B are t-complete then exactly one of the following holds:

Necessary conditions

Theorem

Let $A, B \leq G$ such that $\langle A, B \rangle = G$. If A and B are t-complete then exactly one of the following holds:

- G is t-complete

Necessary conditions

Theorem

Let $A, B \leq G$ such that $\langle A, B \rangle = G$. If A and B are t-complete then exactly one of the following holds:

- G is t-complete
- $G = A \times B$ and $gcd(|A|, |B|) = 1$.

Corollary

Definition

$t(G) := \langle H \leq G \mid H \text{ is t-complete } \rangle$.

Corollary

Definition

$t(G) := \langle H \leq G \mid H \text{ is t-complete } \rangle$.

Properties of $t(G)$

Corollary

Definition

$t(G) := \langle H \leq G \mid H \text{ is t-complete } \rangle$.

Properties of $t(G)$

- $t(G)$ char G.

Corollary

Definition

$t(G) := \langle H \leq G \mid H \text{ is t-complete } \rangle$.

Properties of $t(G)$

- $t(G)$ char G.
- $t(G)$ is the direct product of t-complete maximal subgroups.

Corollary

Definition

$t(G) := \langle H \leq G \mid H \text{ is t-complete } \rangle$.

Properties of $t(G)$

- $t(G)$ char G.
- $t(G)$ is the direct product of t-complete maximal subgroups.
- $t(G) = 1$ implies that G is odd and any two elements of prime order commute.

Homomorphic images

Definition

$T(G) := \bigcap\{N \triangleleft G \mid G/N \text{ is t-complete }\}$.

Homomorphic images

Definition

$T(G) := \bigcap \{ N \triangleleft G \mid G/N \text{ is t-complete } \}.$

Properties of $T(G)$

Homomorphic images

Definition

$T(G) := \bigcap\{N \triangleleft G \mid G/N \text{ is t-complete }\}.$

Properties of $T(G)$

- $T(G)$ char G.

Homomorphic images

Definition

$T(G) := \bigcap\{N \triangleleft G \mid G/N \text{ is t-complete }\}$.

Properties of $T(G)$

- $T(G)$ char G.
- $G/T(G)$ is the subdirect product of pairwise coprime t-complete groups.

Homomorphic images

Definition

$T(G) := \bigcap\{N \triangleleft G \mid G/N \text{ is t-complete }\}$.

Properties of $T(G)$

- $T(G)$ char G.
- $G/T(G)$ is the subdirect product of pairwise coprime t-complete groups.
- For finite G: $T(G) = G$ implies that G is nilpotent of odd order.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.
- Dihedral groups, S_n (for all n), A_n ($n \geq 4$), finite nonabelian simple groups.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.
- Dihedral groups, S_n (for all n), A_n ($n \geq 4$), finite nonabelian simple groups.
- Finite nonabelian groups generated by their elements of prime order that do not have the form $G = A \times B$ with $gcd(|A|, |B|) = 1$.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.
- Dihedral groups, S_n (for all n), A_n ($n \geq 4$), finite nonabelian simple groups.
- Finite nonabelian groups generated by their elements of prime order that do not have the form $G = A \times B$ with $gcd(|A|, |B|) = 1$.
- Groups all of whose minimal normal subgroups are nonabelian.

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.
- Dihedral groups, S_n (for all n), A_n ($n \geq 4$), finite nonabelian simple groups.
- Finite nonabelian groups generated by their elements of prime order that do not have the form $G = A \times B$ with $gcd(|A|, |B|) = 1$.
- Groups all of whose minimal normal subgroups are nonabelian.
- $SL(n, F) \leq G \leq GL(n, F)$ ($n \geq 2$)

t-complete groups

Theorem: Mayr, Horváth, Szabó, Pongrácz

The following groups are t-complete:

- Nonabelian p-groups of exponent p.
- Frobenius groups.
- Dihedral groups, S_n (for all n), A_n ($n \geq 4$), finite nonabelian simple groups.
- Finite nonabelian groups generated by their elements of prime order that do not have the form $G = A \times B$ with $gcd(|A|, |B|) = 1$.
- Groups all of whose minimal normal subgroups are nonabelian.
- $SL(n, F) \leq G \leq GL(n, F)$ ($n \geq 2$)
- Finite perfect groups.

Problems

Subgroup lattices

Problems

Subgroup lattices

- What lattices occur as $L(G)$?

Problems

Subgroup lattices

- What lattices occur as $L(G)$?
- Describe t-complete groups.

Problems

Subgroup lattices

- What lattices occur as $L(G)$?
- Describe t-complete groups.

t(G), T(G)

Problems

Subgroup lattices

- What lattices occur as $L(G)$?
- Describe t-complete groups.

t(G), T(G)

- What are the properties of $t(G)$ and $T(G)$?

Problems

Subgroup lattices

- What lattices occur as $L(G)$?
- Describe t-complete groups.

t(G), T(G)

- What are the properties of $t(G)$ and $T(G)$?
- Characterize groups with $t(G) = 1$.

Problems

Subgroup lattices

- What lattices occur as $L(G)$?
- Describe t-complete groups.

t(G), T(G)

- What are the properties of $t(G)$ and $T(G)$?
- Characterize groups with $t(G) = 1$.
- Characterize groups with $T(G) = G$.