Power representation of modals

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Power algebra

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$\wp(A)$ *- the family of all non-empty subsets of A*

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For any n-ary operation $\omega : A^n \to A$ *we define the complex operation* $\omega : \wp(A)^n \to \wp(A)$ *in the following way:*

$$\omega(A_1, \dots, A_n) := \{\omega(a_1, \dots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \dots, A_n \subseteq A$.

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where $\emptyset \neq A_1, \ldots, A_n \subseteq A$.
The power (complex or global) algebra of an algebra (A, Ω) *is the algebra* $(\wp(A), \Omega)$.

Power algebras of groupoids

Theorem (J.Jezek)

For every groupoid $(A, \cdot)$ there exists an idempotent groupoid $(B, \cdot)$ $(x \cdot x = x)$ such that $(A, \cdot)$ can be embedded into $(\wp(\wp(B)), \cdot)$.

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For every groupoid $(A, \cdot)$ there exists an idempotent groupoid $(B, \cdot)$ $(x \cdot x = x)$ such that $(A, \cdot)$ can be embedded into $(\wp(\wp(B)), \cdot)$. On the other hand, there are groupoids that cannot be embedded into $(\wp(B), \cdot)$ for any idempotent groupoid $(B, \cdot)$.

Varieties of power algebras

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Theorem (G.Gratzer, H.Lakser)

Let $\mathcal{V}$ be a variety. The variety $\wp(\mathcal{V})$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.

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Theorem

The power algebra $(\wp(A), \Omega)$ of an idempotent algebra (A, Ω) is idempotent if and only if each non-empty subset $B \subseteq A$ is a subalgebra of (A, Ω).

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The power algebra $(\wp(A), \Omega)$ *is idempotent if*

- $(A, \cdot)$ *- a left zero-semigroup (groupoid determined by the identity* $xy = x$*)*
- $(A, \cdot)$ *- an equivalence algebra: groupoid with the multiplication defined as follows:*

$$x \cdot y = \begin{cases} x, & \text{if } (x, y) \in \alpha \subseteq A \times A, \\ y, & \text{otherwise} \end{cases}$$

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$\mathcal{V}$ - a variety of idempotent algebras
Idempotent algebras in $\wp(\mathcal{V})$ forms a (proper) subvariety.

Extended power algebras

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The algebra $(\wp_{fin}(A), \Omega, \cup)$ of all finite non-empty subsets of A is a subalgebra of the extended power algebra $(\wp(A), \Omega, \cup)$.
Complex operations distribute over the union $\cup$, i.e. for each n-ary operation $\omega \in \Omega$ and non-empty subsets $A_1, \ldots, A_i, \ldots, A_n, B_i \subseteq A$

$$\omega(A_1, \ldots, A_i \cup B_i, \ldots, A_n) =$$
$$\omega(A_1, \ldots, A_i, \ldots, A_n) \cup \omega(A_1, \ldots, B_i, \ldots, A_n),$$

for any $1 \leqslant i \leqslant n$.

Extended power algebras

Lemma (Monotonicity Lemma)

Let $A_1, \dots, A_n$, $B_1, \dots, B_n$ be non-empty subsets of A and let $\omega \in \Omega$ be an n-ary complex operation over A. If $A_i \subseteq B_i$ for each $1 \leqslant i \leqslant n$, then $\omega(A_1, \dots, A_n) \subseteq \omega(B_1, \dots, B_n)$.

Lemma (Convexity Lemma)

Let $\emptyset \neq A_{ij} \subseteq A$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$. Then

$$\omega(A_{11}, \dots, A_{n1}) \cup \dots \cup \omega(A_{1r}, \dots, A_{nr}) \subseteq$$

$$\omega(A_{11} \cup \dots \cup A_{1r}, \dots, A_{n1} \cup \dots \cup A_{nr}),$$

for each n-ary complex operation $\omega \in \Omega$.

Modals

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Definition

An algebra (M, Ω) *is entropic if any two of its operation commute.*

Examples of modals

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- the algebra $(\mathbb{R}, \underline{I}^0, max)$ defined on the set of real numbers, where $\underline{I}^0$ is the set of the following binary operations:

$$\underline{p} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \;\; (x, y) \mapsto (1 - p)x + py,$$

for each $p \in (0, 1) \subset \mathbb{R}$

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Lemma (Sum-Superiority Lemma)

For each n-ary basic operation $\omega \in \Omega$ *and elements* $x_1, \ldots, x_n \in M$, *one has*

$$\omega(x_1, \ldots, x_n) \leqslant x_1 + \ldots + x_n.$$

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Theorem

Let (A, Ω) be an idempotent algebra. The power algebra $(\wp(A), \Omega)$ is idempotent if and only if for each n-ary basic operation $\omega \in \Omega$ and subsets $A_1, \ldots, A_n \in \wp(A)$

$$\omega(A_1, \ldots, A_n) \subseteq A_1 \cup \ldots \cup A_n.$$

Power algebras of modes

The entropic law may also be expressed by means of (linear) identities:

$$\omega(\phi(x_{11},\ldots,x_{n1}),\ldots,\phi(x_{1m},\ldots,x_{nm})) =$$

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for every n-ary $\omega \in \Omega$ and m-ary $\phi \in \Omega$.

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Power algebras of modes preserve entropic law, but very rarely they are again modes.
Extended power algebras of modes needn't be modals.

Congruences of the extended power algebra

ρ, α- congruences of the extended power algebra $(\wp(M), \Omega, \cup)$ of a mode (M, Ω):

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$$X \;\rho\; Y \quad \Leftrightarrow \quad \text{there exist terms } t \text{ and } s \text{ such that}$$
$$X \subseteq t(Y, Y, \ldots, Y) \quad \text{and} \quad Y \subseteq s(X, X, \ldots, X)$$

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The relation ρ is the least element in the set $Con_{id}(\wp(M))$, of all congruence relations γ on the extended power algebra $(\wp(M), \Omega, \cup)$, such that the quotient $(\wp(M)/\gamma, \Omega)$ is idempotent.

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$$\rho_{|\wp_{fin}(M)} = \alpha_{|\wp_{fin}(M)}$$

Representation theorem for modals

Theorem

Let (M, Ω) be a mode. The quotient algebra $(\wp(M)/\alpha, \Omega, \cup)$ is isomorphic to the modal $(MS, \Omega, +)$ of all non-empty subalgebras of (M, Ω) and the quotient algebra $(\wp_{fin}(M)/\alpha, \Omega, \cup)$ is isomorphic to the modal $(MP, \Omega, +)$ of all finitely generated subalgebras.

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Theorem (Power representation Theorem)

Let $(M, \Omega, +)$ be a modal generated by a set X. Then $(M, \Omega, +) \in HS(\wp(\langle X \rangle_\Omega), \Omega, \cup)$.

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Theorem (Power representation Theorem)

Let $(M, \Omega, +)$ *be a modal generated by a set* X*. Then* $(M, \Omega, +) \in HS(\wp(\langle X \rangle_\Omega), \Omega, \cup)$.

Corollary

Each modal $(M, \Omega, +)$ *generated by a set* X *is a homomorphic image of* $(\langle X \rangle_\Omega P, \Omega, +)$.

Thank you for your attention!