# RING CLOSURE

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Consider the category of associative rings with unity and ring morphism that preserves unity, denoted by $R1NG$. The modules over a ring $R$ will be *unitary* and left modules, its corresponding category is denoted by $R-Mod$.

$R1NG$ can be thought of as a class with a preorder given by $R, S \in R1NG, R \lesssim S$ if there exists a monomorphism $f: R \to S$, this gives a partial order in the class. For example, the algebraic closure of the field of rational functions defined over $\mathbb{Q}$ and in $\varkappa_0$ indeterminate and the field of rational functions in an indeterminate over the previous field.

So this doesn't provide antisymmetry in the sense of an isomorphism, so when $R \lesssim S$ y $S \lesssim R$ it's denoted $R \approx S$.

A ring closure is a functor $\overline{\phantom{R}} : R1NG \to R1NG$, which meets that for all $R, S \in R1NG$,

(1. Extensive) $R \lesssim \bar{R}$

(2. Monotone) If $R \lesssim S$ then $\bar{R} \lesssim \bar{S}$

(3. Idempotent) $\bar{R} \approx \bar{\bar{R}}$

The simplest example of ring closure is the $\varkappa_0$ −polynomial closure defined as $\bar{R} = R[\varkappa_0]$.

The second example is the matricial closure. For a ring,

$$R_n = M_{2^n}(R)$$

with $R_0 = R_1$ and the sequence of morphisms

$$\{\alpha_n : R_n \to R_{n+1}\}_{n\in\mathbb{N}}$$

$$\alpha_n(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

Define recursively

$$\alpha_i^i = 1_{R_i}$$

$$\alpha_i^{i+j+1} = \alpha_j^{j+1}\alpha_i^j$$

This gives a directed system, so the matricial closure is defined recursively by

$$\bar{R} = \lim_{\rightarrow} R_n$$

There are more ways to get this construction, like taking $K^n \times K^n$ matrices or the partial order induced by division.

Here is where we notice that both previous closures are nothing more than direct limit of overfunctors of the identity, that is, consider an overfunctor $F$ of the identity, which is a functor for which $R \hookrightarrow F(R)$ defines recursively

$$F^0(R) = R$$

$$F^{n+1}(R) = F^n\big(F(R)\big)$$

And in the natural way a guided system is constructed and is called $F = \lim_{\rightarrow} F$.

Now $\bar{F}$ is a ring closure, include that this construction can be done in any category with direct limits.

Now the two previous mentioned come from the $\_[X]$ overfunctors, a polynomial functor variable $a$ and $M_2(\_)$ $2-$matricial functor, this closures are denoted $\_[\varkappa_0]$ y $\bar{M}(\,)$.

The observation is that the functors $\_[x_1, \ldots, x_m]$ y $M_n(\,)$ also induce the same closures for any $n \in \mathbb{N}$.

Not every closure can be obtained this way, in the sense of being a strictly crescent limit, like the $\kappa$ −polynomial closure with $\kappa > \varkappa_0$.

Another example would be for a given group $G$, consider the overfunctor $\_[G]$ and call its closure $\_\overline{[G]}$, when $G$ is an abelian group $\_[G^{\mathbb{N}}]$ stops in the first step.

The natural question is if $R$ meets a property, $\bar{R}$ does too?

That’s too say which properties preserve under closure.

For example, the polynomial closure in $\kappa$ variables preserves the Goldie’s dimension if $G$ is a free group.

If $R$ has Goldie’s finite dimension then $R\left[G^{(\mathbb{N})}\right]$ will also have Goldie’s finite dimension.

If $R$ is Von Neumann regular, i.e., for each $a \in R$ exists $\in R$ $a = axa$, then $\bar{M}(R)$ is too.

But now surges a strange question, which properties are not preserved?

The reason of this question is simple, if we call a ring closed if $\bar{R} \approx R$, every closed ring shall be obtained by applying ring closure to a ring which is not closed? Will it be minimal in this sense, for example, for the matricial closure it doesn't satisfy

a) Artinian

b) Noetherian

c) Commutative

d) Field

e) Boolean

f) Integer domain

In fact, if $R$ has cardinality less than $2^{\aleph_0}$ there will be at least $2^{\aleph_0}$ simple modules which are not isomorph.

Another simple thing to see is that if $R$ is closed then it has Goldie's infinite dimension.

The question that remains is that if given a closed ring, it will always come from a ring which is not closed and if this will be minimal.

THEOREM. The matricial closure of an artinian ring is semiartinian.

# THANKS!