Weakly oligomorphic clones

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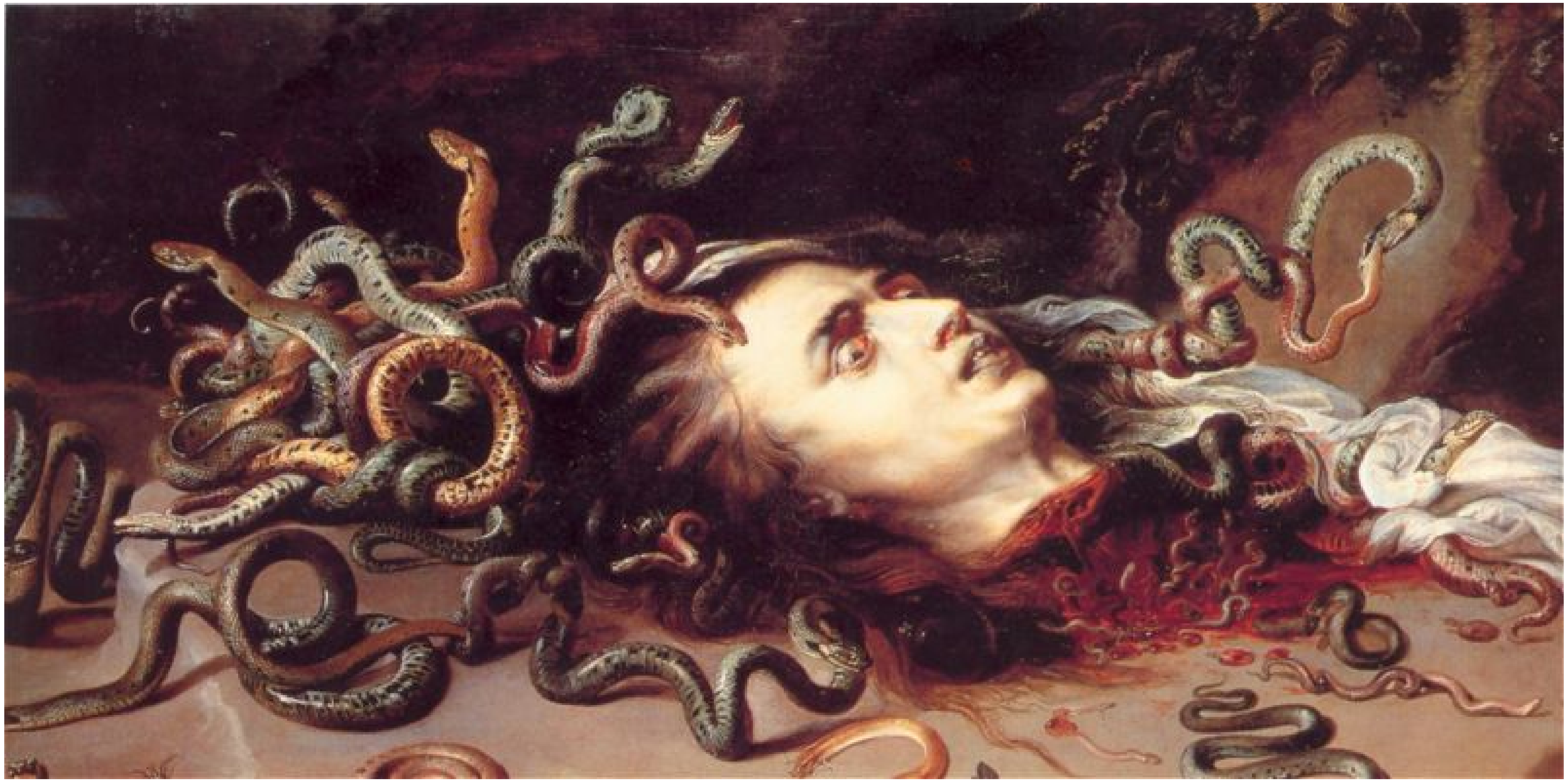
The lattice of clones on infinite sets



(Rubens, *The Head of Medusa*. Kunsthistorisches Museum, Vienna.)

The lattice of clones on infinite sets

How to fight the Medusa?

- don't fight
- specialize (e.g. to locally closed clones)
- subspecialize (e.g. to special locally closed clones)

☞ Goldstern, M., Pinsker, M., *A survey of clones on infinite sets*. Algebra univers. 59 (2008) 365–403

Subspecialization: Oligomorphic clones

A permutation group $G \leqslant \mathrm{Sym}(A)$ on a countable set A is *oligomorphic* if $\mathrm{Inv}^{(k)} G$ is finite, for all $k \geqslant 1$.

Theorem. (Engeler, Ryll-Nardzewski, Svenonius)
A countable relational strucutre (A, Γ) is ω-categorical if and only if $\mathrm{Aut}(A, \Gamma)$ *is an oligomorphic permutation group.*

Subspecialization: Oligomorphic clones

Definition. A locally closed clone C on a countable set A is *oligomorphic* if

- $\mathrm{G} = \mathrm{C} \cap \mathsf{Sym}(A)$ is a group, and
- G is oligomorphic.

Question. Which parts of universal algebra can be generalised from clones over a finite domain to oligomorphic clones?

(Bodirsky, M., Chen, H., *Oligomorphic clones*. Algebra universalis 57 (2007), 109–125)

Subspecialization: Oligomorphic clones

Many beautiful properties, e.g.

- if Γ is an ω-categorical structure, then Pol Γ contains a minimal oligomorphic clone;
- there are four types of minimal oligomorphic clones that closely resemble the structure of minimal clones on a finite set;
- a generalization of the Baker-Pixley theorem.

(Bodirsky, M., Chen, H., *Oligomorphic clones*. Algebra universalis 57 (2007), 109–125)

Subspecialization: Oligomorphic clones

Moreover,

Theorem. *Let* Γ *be an* ω*-categorical structure. Then*

$$\langle \Gamma \rangle_{\mathrm{pp}} = \mathrm{Inv}\,\mathrm{Pol}\,\Gamma.$$

(M. Bodirsky, J. Nešetřil, *Constraint Satisfaction with Countable Homogeneous Templates*, Journal of Logic and Computation 2006 16(3):359-373)

However...

Maximal oligomorphic clones?

Oligomorphic clones are scattered accross the poset of locally closed clones:

> if C is an oligomorphic clone and $\mathrm{C} \subseteq \mathrm{D}$, D need not be an oligomorphic clone simply because $\mathrm{D} \cap \mathsf{Sym}(A)$ need not be a group.

Weakly oligomorphic clones

Clones are more intimately related to transformation monoids than to permutation groups.

Definition. A transformation monoid $M \leqslant A^A$ on a countable set A is *oligomorphic* if $\mathrm{Inv}^{(k)} M$ is finite, for all $k \geqslant 1$.

(D. Mašulović, M. Pech, *Oligomorphic transformation monoids and homomorphism-homogeneous structures*, submitted)

Weakly oligomorphic clones

Definition. A locally closed clone C is *weakly oligomorphic* if $C^{(1)}$ is an oligomorphic transformation monoid.

Theorem. *The following are equivalent for a clone* C *on an infinite set:*

- C *is a weakly oligomorphic clone;*
- $\mathrm{Inv}^{(k)} C^{(m)}$ *is finite for all* $k, m \geqslant 1$*;*
- $\mathrm{Inv}^{(k)} C$ *is finite for all* $k \geqslant 1$*;*
- $\mathcal{M}_k(C)$ *is an oligomorphic transf. monoid for all* $k \geqslant 1$*.*

Weakly oligomorphic clones

Proposition. *Every oligomorphic clone is weakly oligomorphic. The converse is not true.*

Theorem. *Let* Γ *be a weakly oligomorphic structure. Then*

$$\langle \Gamma \rangle_{\mathsf{pp}} = \mathsf{Inv}\,\mathsf{Pol}\,\Gamma.$$

$$\boxed{\text{Weakly oligo clones}} \quad \underset{\mathsf{Inv}}{\overset{\mathsf{Pol}}{\rightleftarrows}} \quad \boxed{\text{Weakly oligo structures}}$$

Weakly oligomorphic clones

Proposition. *The poset of weakly oligomorphic clones is upward closed, that is, if* C *is a weakly oligomorphic clone and* $\mathrm{C} \subseteq \mathrm{C}_1$*, then* C_1 *is also a weakly oligomorphic clone.*

Proposition. *Let* A *be a countable set. The poset of weakly oligomorphic clones on* A *is not dual atomic. Consequently, there exist towers of weakly oligomorphic clones.*

Weakly oligomorphic clones

Some maximal locally closed clones are weakly oligomorphic (and hence maximal in the poset of weakly oligomorphic clones).

Proposition. *Let A be a countable set.*

- *Both maximal clones that contain A^A are weakly oligo.*
- *If $\varnothing \neq \rho \subset A$, then* $\operatorname{Pol} \rho$ *is weakly oligo.*
- *If $\rho \subseteq A^2$ is a central relation, then* $\operatorname{Pol} \rho$ *is weakly oligo.*
- *If ρ is a nontrivi. equiv. relation, then* $\operatorname{Pol} \rho$ *is weakly oligo.*

Weakly oligomorphic clones

Proposition. *Let $\mathbb{A}$ be a countable set, and let $\leqslant$ be a locally bounded partial order on $\mathbb{A}$. If $(\mathbb{A}, \leqslant)$ is homomorphism-homogeneous then* $\mathrm{Pol} \leqslant$ *is weakly oligo.*

Problem 1. Let A be a countable set. Characterize locally bounded partial orders $\leqslant$ on A with the property that $\mathrm{Pol} \leqslant$ is weakly oligo.

Problem 2. Let A be a countable set. Characterize maximal locally closed clones which are is weakly oligo.

Thank you