A personal history of tournaments represented as groupoids

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Müller, Nešetřil and Pelant: Either tournaments or algebras? (Discrete Math, 1975) : Is the variety finitely based? An infinite independent set of identities is given.

How I heard of tournaments

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- A political fight in a high school has strange consequences

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- A political fight in a high school has strange consequences
- "I think I heard of the middle guy"

Basic tools

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Birkhoff's theorem

A locally finite variety $\mathcal{V}$ is finitely based iff $\mathcal{V} = \mathcal{V}^k$ for some k.

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Represent algebras in the variety as graphs
Also: Coffee, cigarettes and brains.

Advanced tools

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None work for this problem.

The construction

SI groupoid which is almost a tournament.

The construction

SI groupoid which is almost a tournament. Using whiteboard as I don't know how to draw in LaTeX.

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Theorem

The variety generated by tournaments is not finitely based.

The end?

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So, should we stop here?

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So, should we stop here? After all, we solved the problem.

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Problem

Is every finite tournament finitely based?

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Willard's theorem (improved version by Kearnes and Willard)

A locally finite variety $\mathcal{V}$ which generates a congruence meet-semidistributive residually [strictly] finite variety is finitely based.

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Does every finite tournament generate a residually finite variety?

Subdirectly irreducible finite tournaments

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Have to go to whiteboard again,

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Problem

Is the variety generated by tournaments equal to the quasivariety generated by tournaments?

Partial results

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Tools needed: Coffee, cigarettes, brains and a little tame congruence theory.

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Tools needed: Jarda-style persistence.

Full result

Triangular ideals

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Back to the whiteboard.

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Maróti's theorem - from his PhD thesis

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- When all else fails, try syntax.

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Another Birkhoff's theorem

A locally finite variety $\mathcal{V}$ is inherently nonfinitely based iff $\mathcal{V}^k$ is not locally finite for all k.

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Connections with CSP

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2-semilattices have a finite relational width.

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Bulatov's theorem

2-semilattices have a finite relational width.

This is the nicest bounded width proof (as NU is too easy and special) and a template for many other which followed (CD(3) and CD(4), for example), till Barto and Kozik had to invent a lot of completely new ideas to tackle the general case.

Connections with CSP - continued

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Another Bulatov's theorem

The CSP dichotomy holds for conservative algebras.
Equivalently, the list-homomorphism problem admits dichotomy.

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Lemma

Let $\mathbf{A} \leq_{sd} \mathbf{B} \times \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are strongly connected algebras in $\mathcal{T}$. If there exists $c \in C$ such that $B \times \{c\} \subseteq A$, then $A = B \times C$.

- J. Ježek, Constructions over tournaments, Czech. Math. J., Vol. 53, no. 2 (2003) 413–428.
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Thanks for your attention

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JARDA, GET WELL SOON!