Second look at cyclic terms

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- a Taylor term if it is idempotent and, for any $j \leq n$;

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Theorem (Maróti and McKenzie)

Let $\mathcal{V}$ be a locally finite variety then TFAE:

- *$\mathcal{V}$ has a Taylor term;*
- *$\mathcal{V}$ has a weak near-unanimity term.*

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Theorem (Maróti and McKenzie)

Let f be an n-ary function on a finite set satisfying identities of a Taylor term. By composing and identifying coordinates a function satisfying the weak near-unanimity identities can be produced from f.

A term $t(x_1, \ldots, x_n)$ is

- cyclic if it is idempotent and $t(x_1, \ldots, x_n) \approx t(x_2, \ldots, x_n, x_1)$.

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Theorem (Barto, Kozik)

For a finite algebra $\mathbf{A}$ *TFAE:*

- $\mathbf{A}$ *has a Taylor term;*
- $\mathbf{A}$ *has a cyclic term;*
- $\mathbf{A}$ *has a cyclic term of arity* p*, for every prime* $p > |A|$*.*

Part I

We start slowly:

Lemma

Let $\mathbf{A}$ be a finite idempotent algebra. Then there exists a term t such that for any $B \subseteq A$ and any $b \in \mathrm{Sg}_{\mathbf{A}}(B)$ there exists $b_1, \ldots, b_n \in B$ such that $t(b_1, \ldots, b_n) = b$.

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- the term $t(x_1, \ldots, x_n)$ works for (B, c) if there are $b_1, \ldots, b_n \in B$ such that

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- for two terms $t(x_1, \ldots, x_n)$ and $s(x_1, \ldots, x_m)$ the term

$$s(t(x_1, \ldots, x_n), \ldots, t(x_{nm-n+1}, \ldots, x_{nm}))$$

works for (B, c) given $t(x_1, \ldots, x_n)$ or $s(x_1, \ldots, x_m)$ work for (B, c).

Definition (VBD-absorbing subalgebra)

Let **A** *be a finite idempotent algebra. The subalgebra* **B** $\leq$ **A** *is VBD-absorbing if there exists a term* $t(x_1, \ldots, x_n)$ *such that*

$$t(a_1, \ldots, a_n) \in B \text{ whenever } \{a_1, \ldots, a_n\} \cap B \neq \emptyset$$

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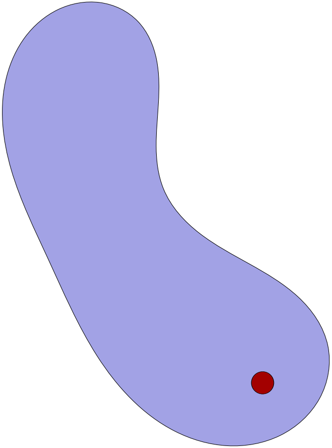
Lemma (Barto)

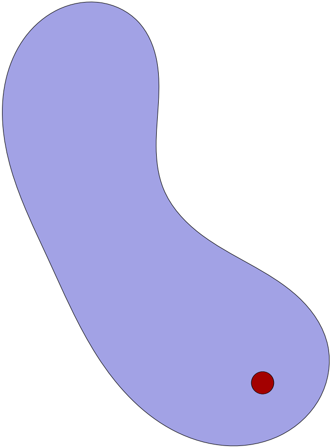
Let $\mathbf{A}$ be a finite idempotent algebra with a Taylor term then:

- $\mathbf{A}$ has a proper VBD-absorbing subalgebra, or
- there is a term $t(x_1, \ldots, x_n)$ (a magic term) such that, for any $b, c \in A$ and any $j \leq n$ there are $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ such that:

$$t(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) = c.$$



- blue is a subuniverse that can be obtained from $s(x_1, \ldots, x_n)$ with b at position j;



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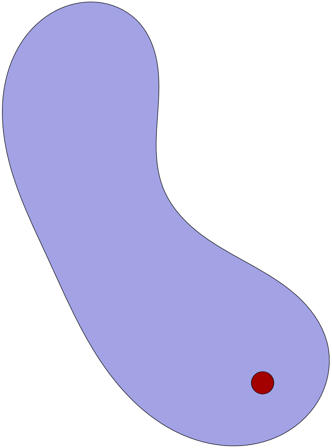

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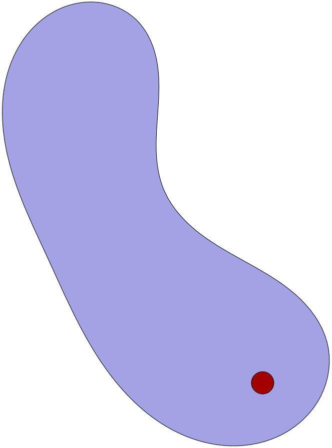

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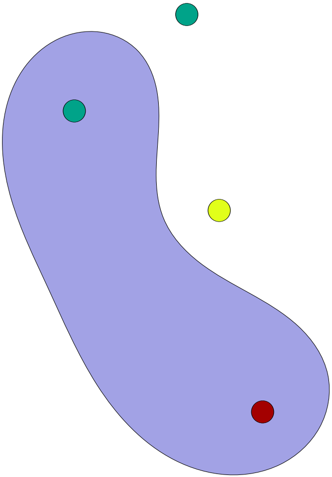
- $T_1(x, y) := T(x, \ldots) \approx T(y, \ldots)$



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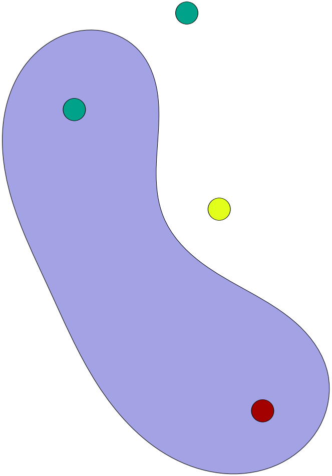

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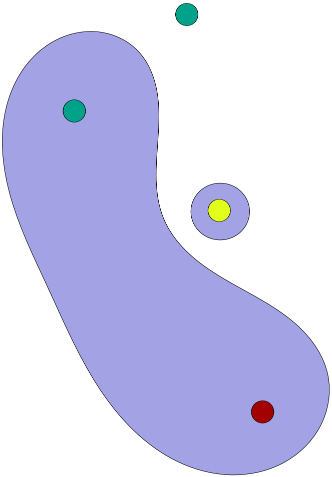

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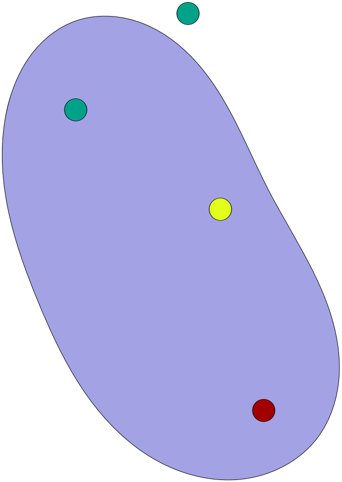
- but if b is not blue then d can be obtained as well since $T_1(b, c) = T(c, \dots)$



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- but if b is not blue then d can be obtained as well since $T_1(b, c) = T(c, \dots)$
- using previous lemma we can obtain a bigger subuniverse.

Definition (Absorbing subalgebra)

Let $\mathbf{A}$ *be a finite idempotent algebra. The subalgebra* $\mathbf{B} \leq \mathbf{A}$ *is absorbing (and write* $\mathbf{B} \lhd \mathbf{A}$*) if there exists a term* $t(x_1, \ldots, x_n)$ *such that*

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A set $R \subseteq A \times B$ *is linked if* $\underbrace{R \circ R^{-1} \circ R \circ \cdots \circ R^{-1}}_{n} = B^2$ *for some* n*.*

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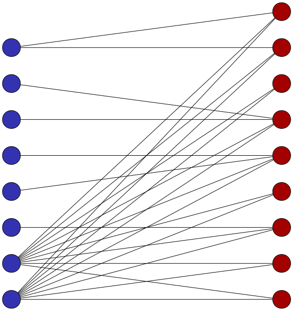
Theorem (Absorption theorem)

Let $\mathbf{A} \leq_s \mathbf{B} \times \mathbf{C}$ *be algebras with a Taylor term, and let* $A \subseteq B \times C$ *be linked. Then:*

- $\mathbf{A} = \mathbf{B} \times \mathbf{C}$, *or*
- $\mathbf{B}$ *or* $\mathbf{C}$ *has a proper absorbing subalgebra*

Special case:



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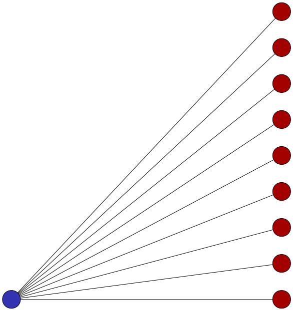

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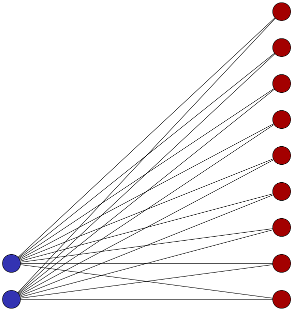

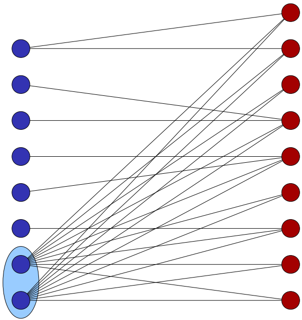

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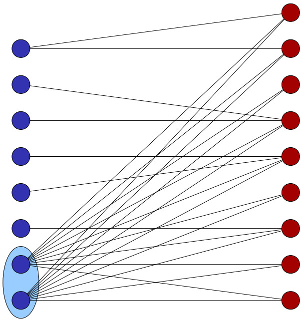

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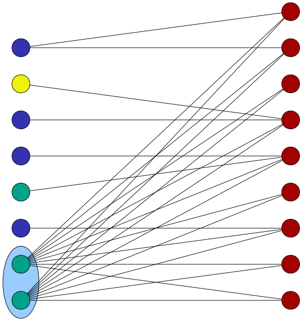
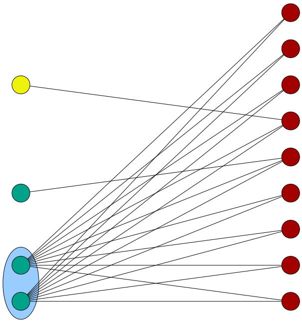


Special case:

- elements that arrow everything on red side are blue
- blue is a subuniverse of blue side
- blue is not absorbing so $t(a_1, a_2, a_3) = a_4$ for the magic term $t(x_1, x_2, x_3)$.



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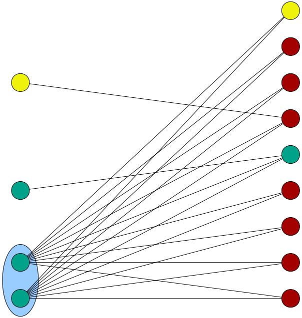

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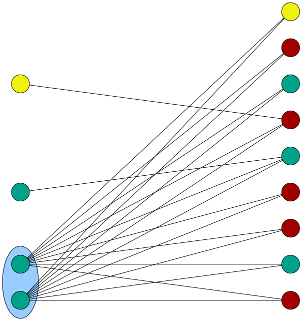

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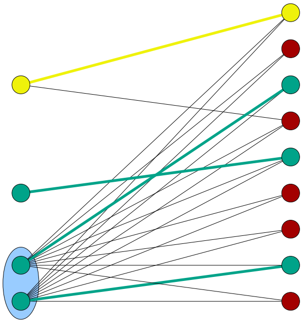
- by lemma we can find b_2 and b_3 s.t. $t(b_1, b_2, b_3) = b_4$



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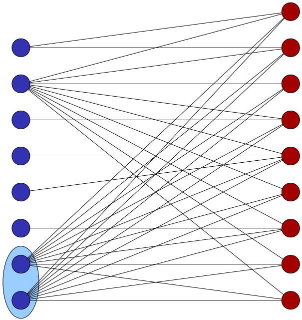
- b_1 is fixed and b_4 is arbitrary both on the red side
- by lemma we can find b_2 and b_3 s.t. $t(b_1, b_2, b_3) = b_4$
- this implies edge from a_4 to b_4



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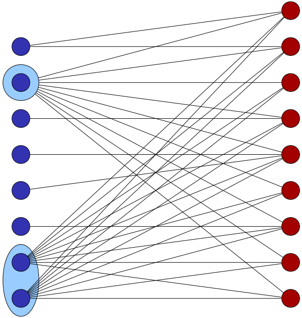
- since b_4 was arbitrary we get more edges



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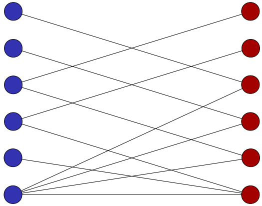
- since b_4 was arbitrary we get more edges
- and can extend blue



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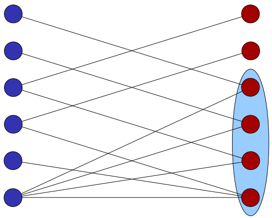

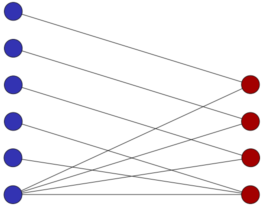

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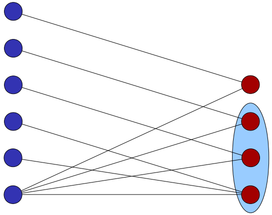

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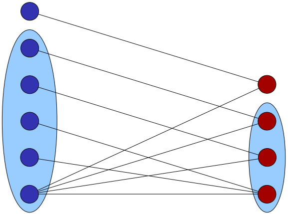

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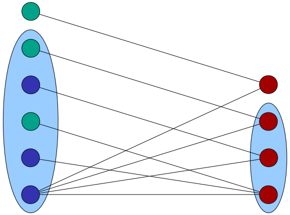

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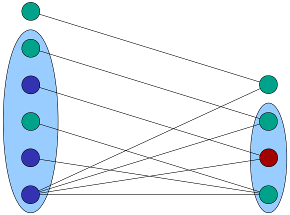

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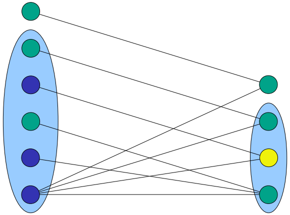

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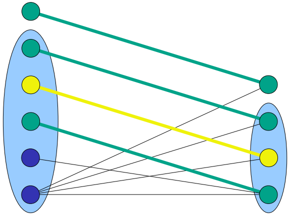

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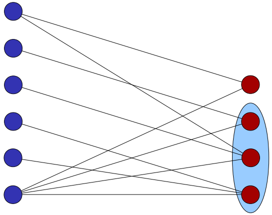

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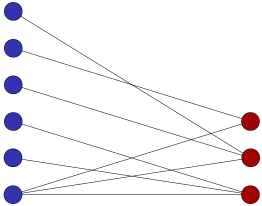

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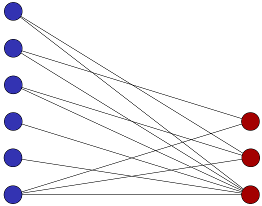

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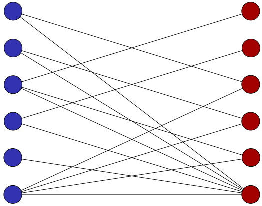

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- looking from right to left we have a situation from simple case again and we are done

Part II

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But sometimes we need a more specific b...

to be more specific:

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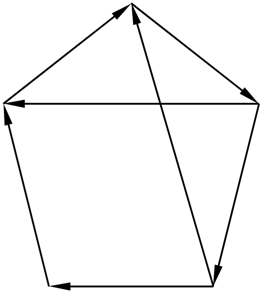
- *in some minimal absorbing subuniverse in such a component (if there is one).*

Case of an absorbing set (connected):



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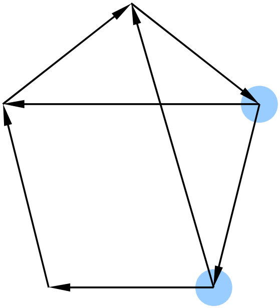

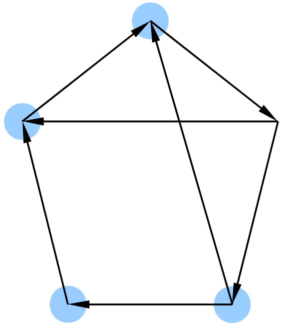

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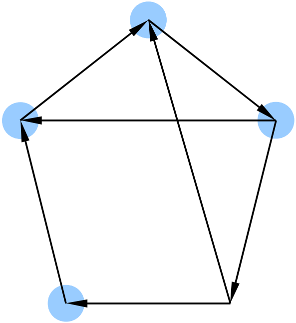
- so is its forward neighbourhood



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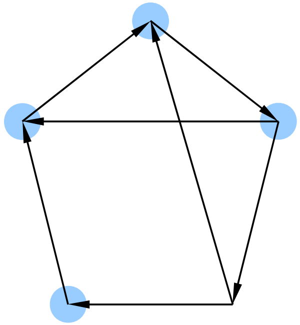

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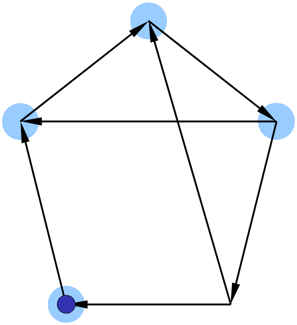

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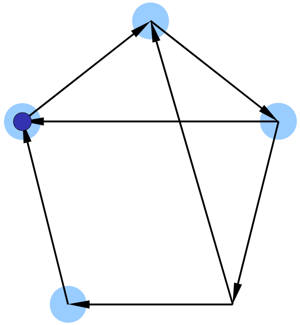

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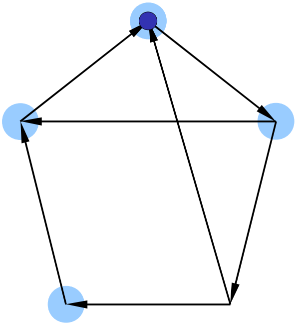

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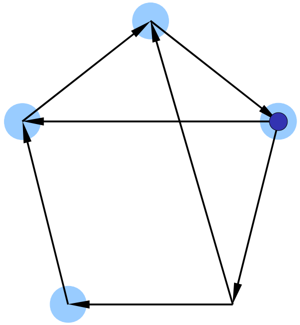

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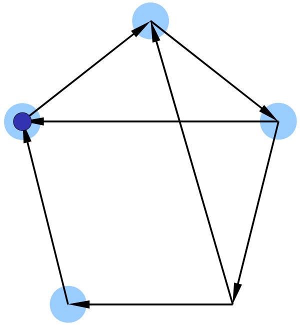

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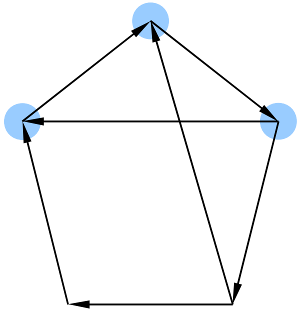

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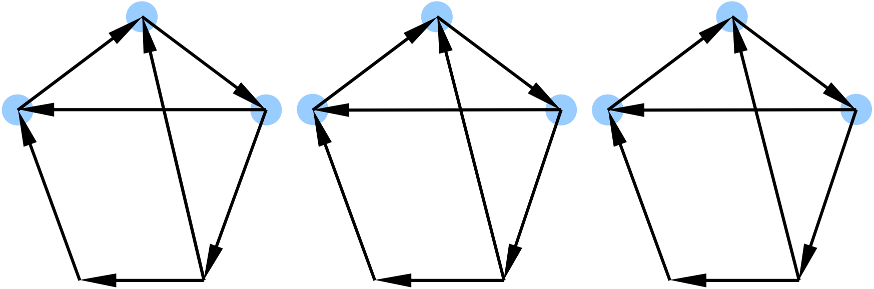
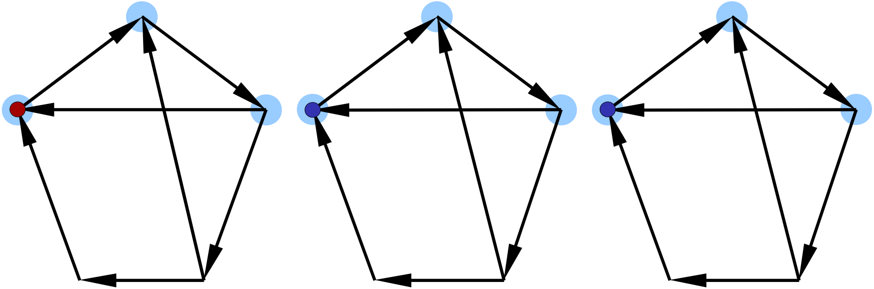
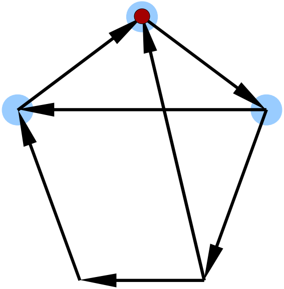
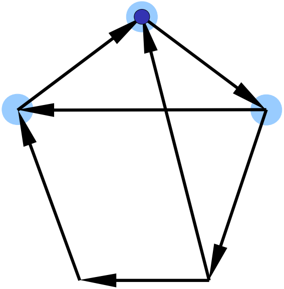
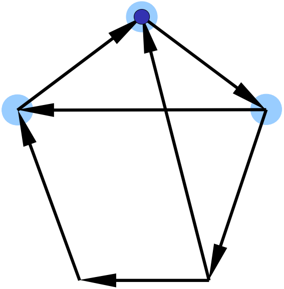
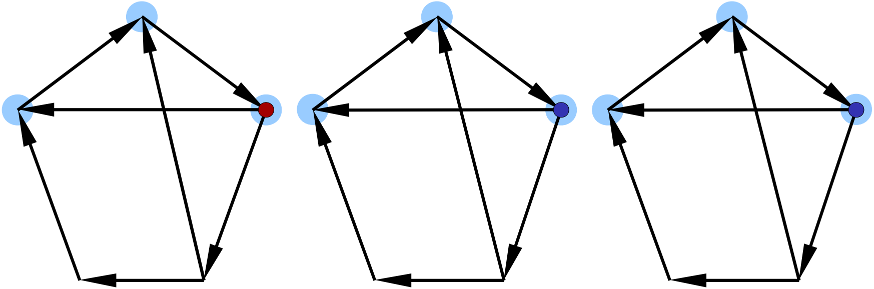
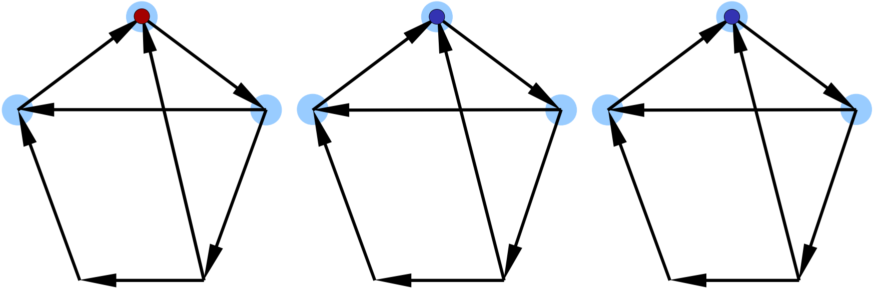
- all elements in smooth graph inside blue form a new and better blue

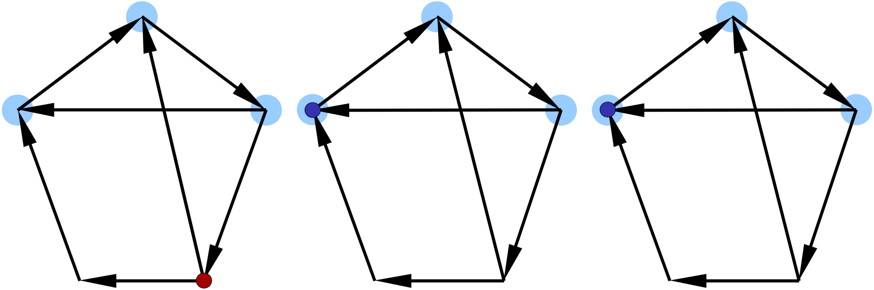


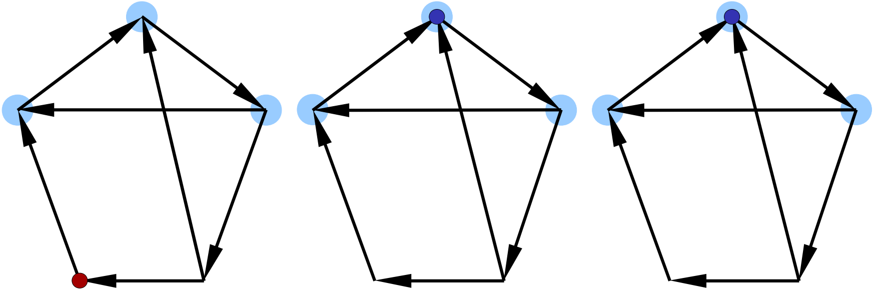


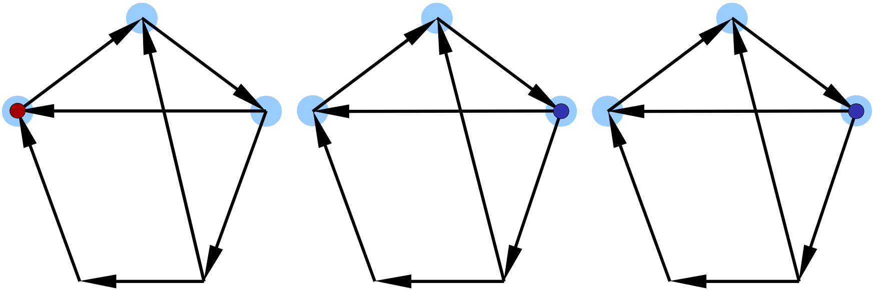




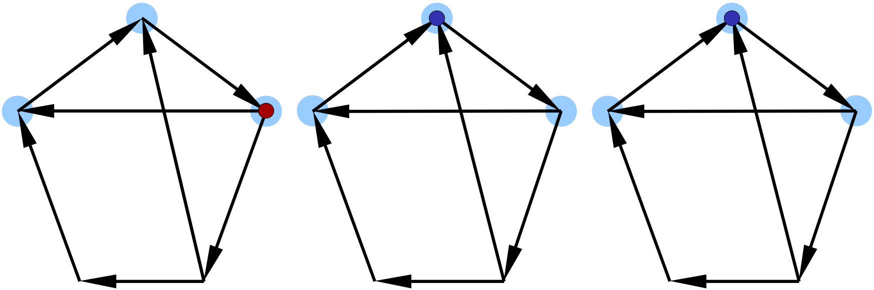


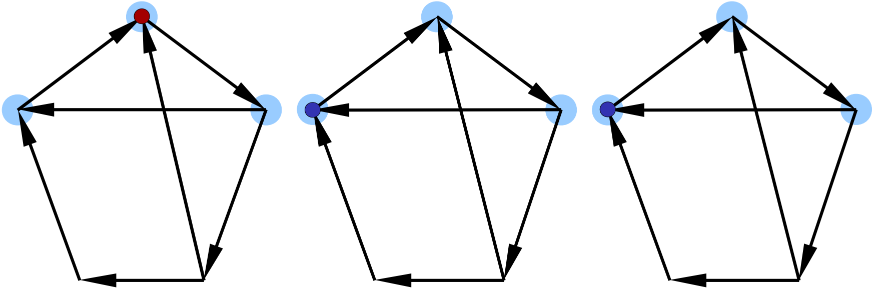


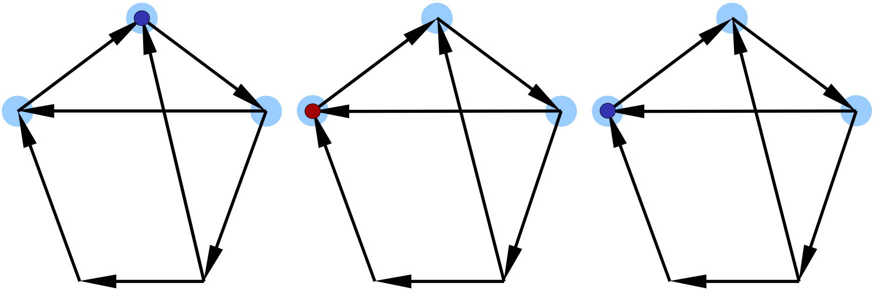


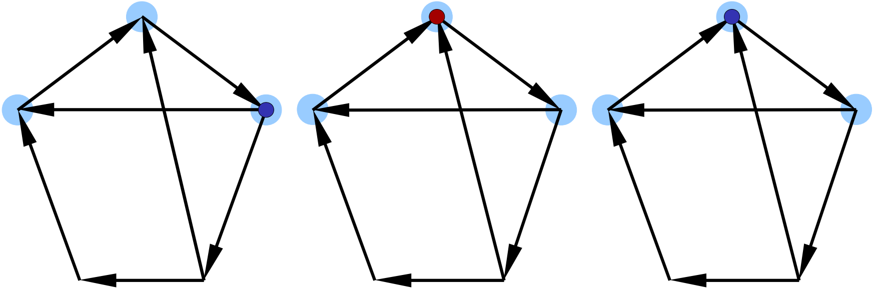


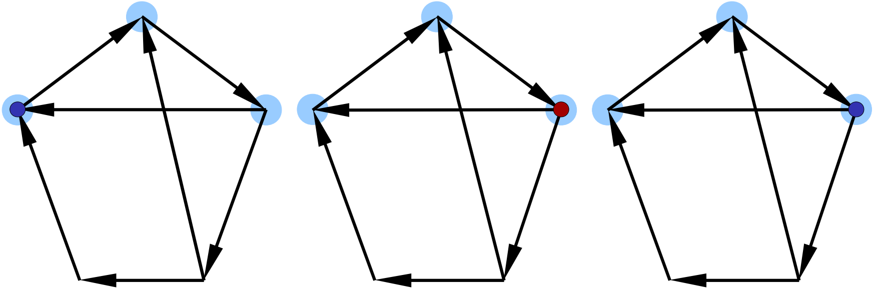


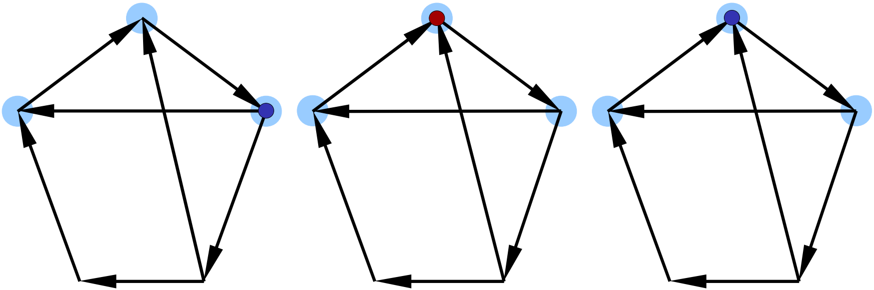


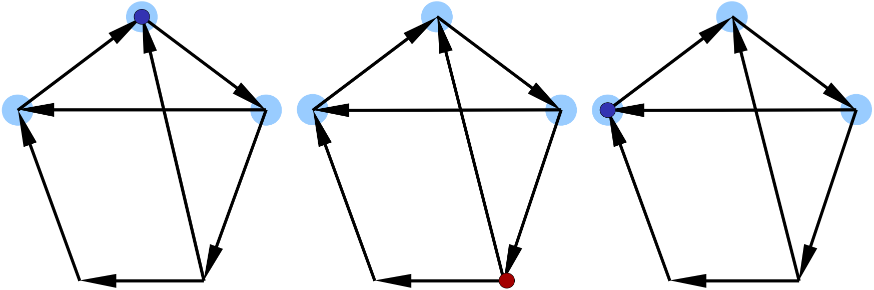


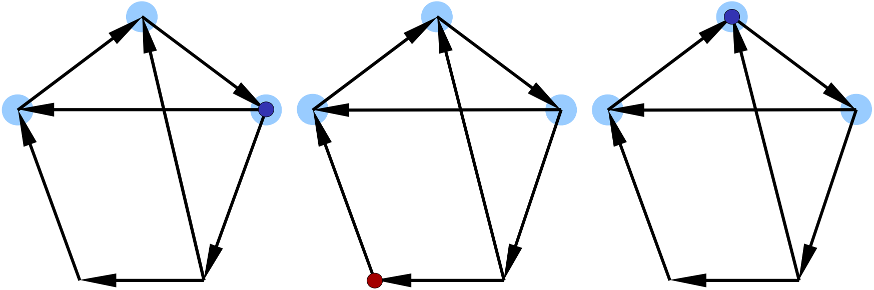


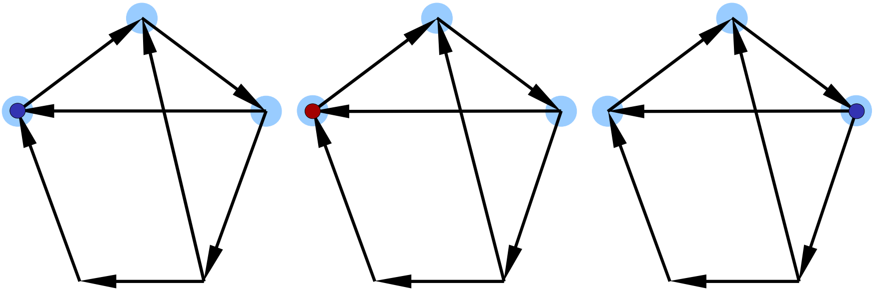


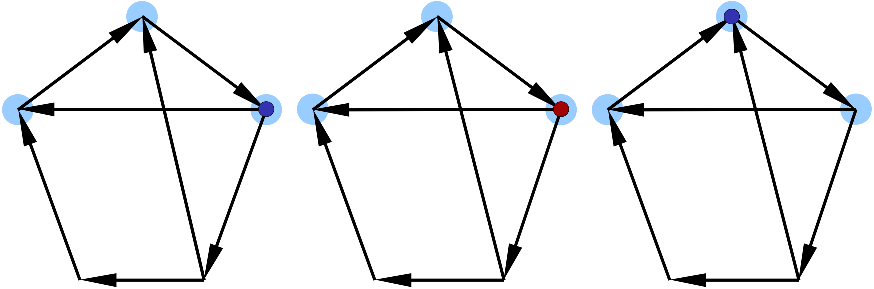


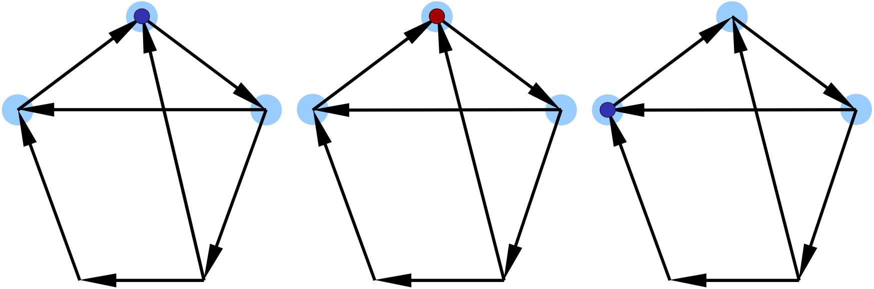


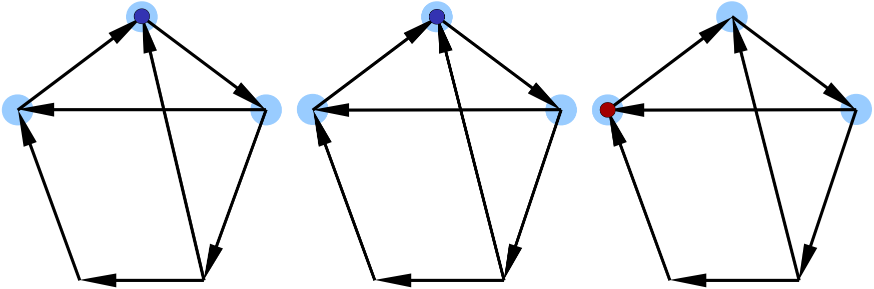


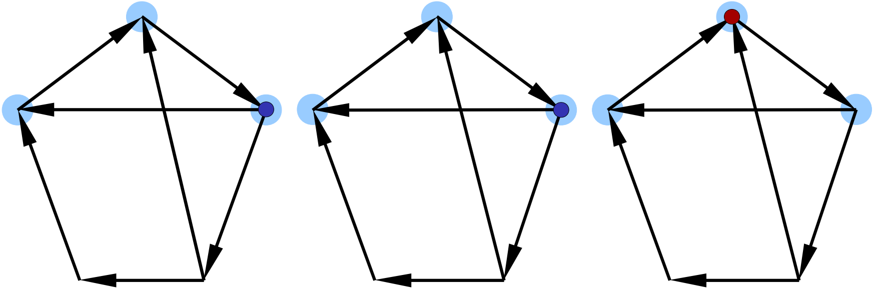


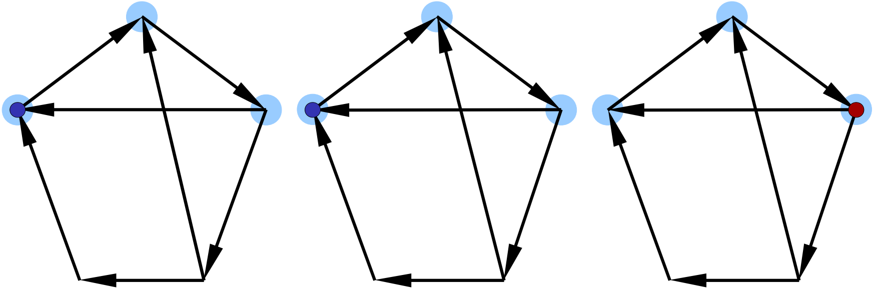


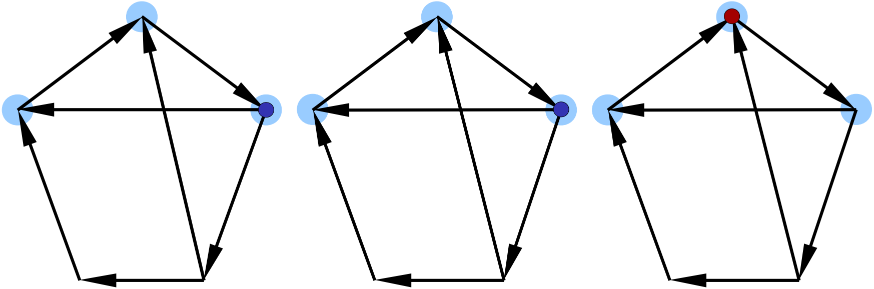


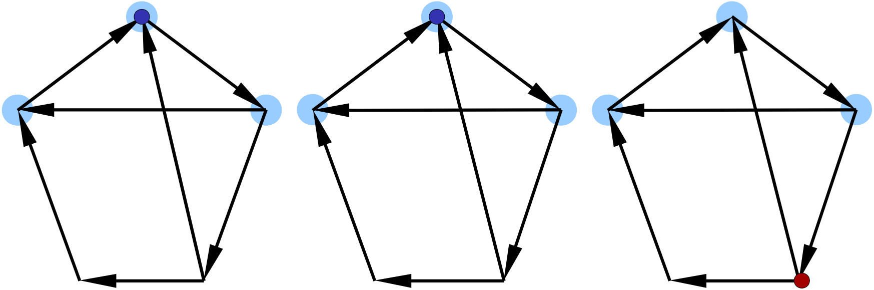


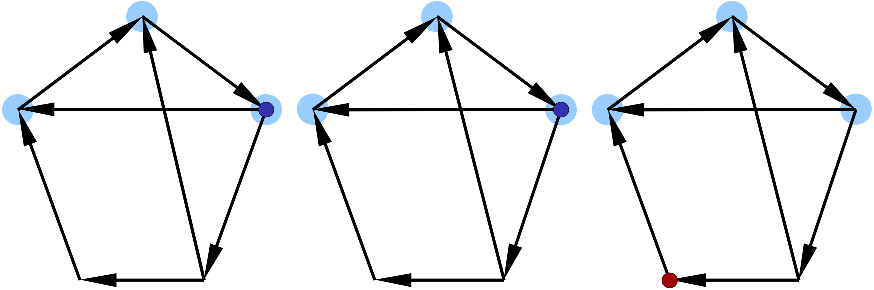


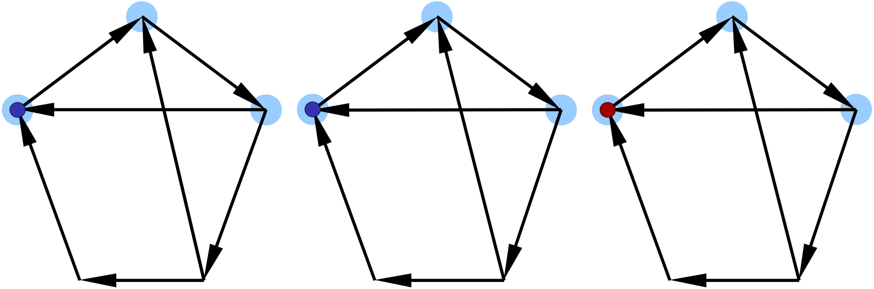


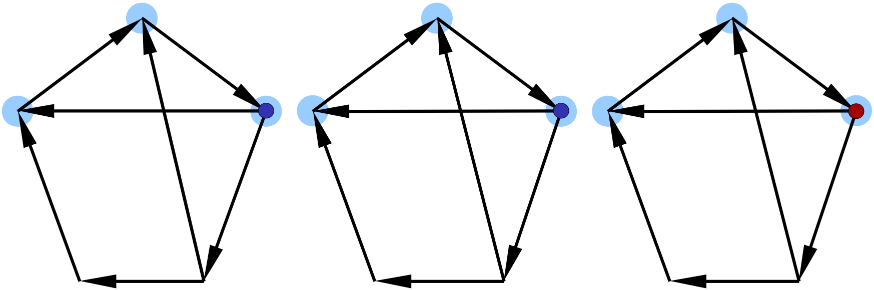


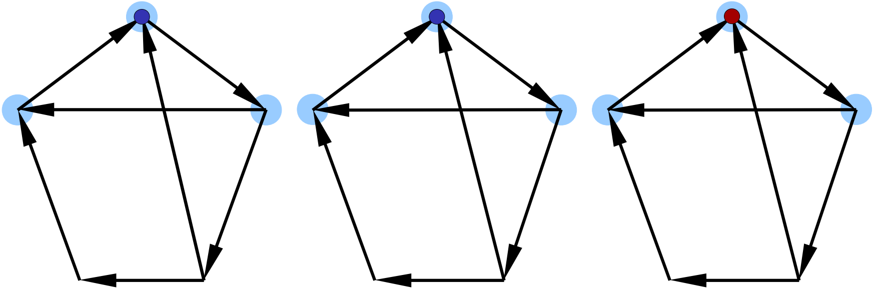






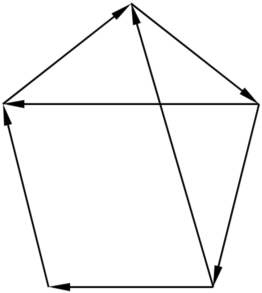






Case of an no absorbing set (connected):

- if E is the set of edges, then $E \circ E$ is dark blue



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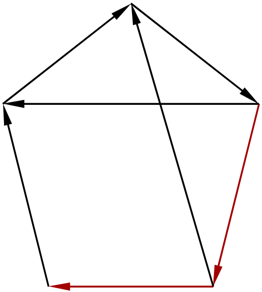

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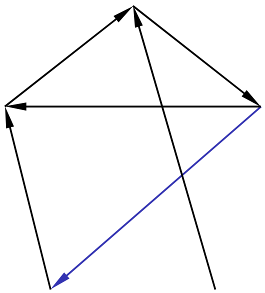

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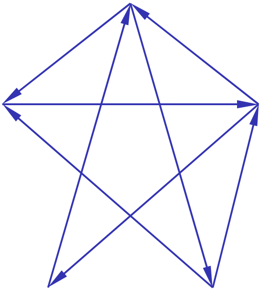

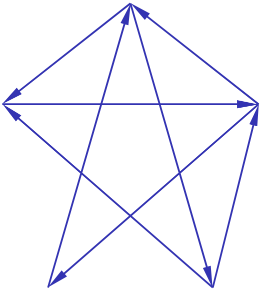

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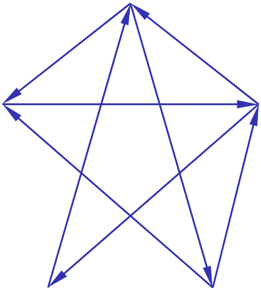

Case of an no absorbing set (connected):

- if E is the set of edges, then $E \circ E$ is dark blue
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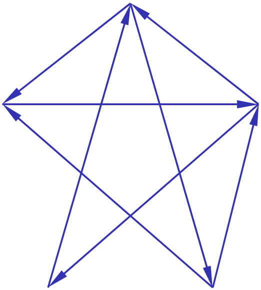

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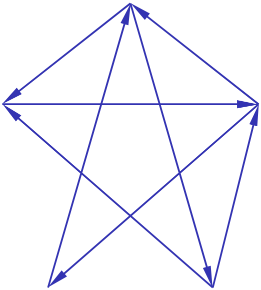

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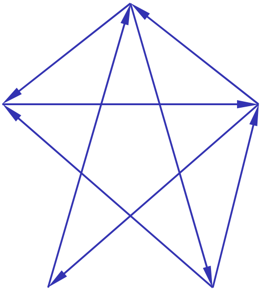

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- and it is not the full congruence

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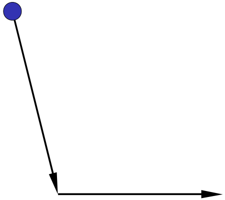

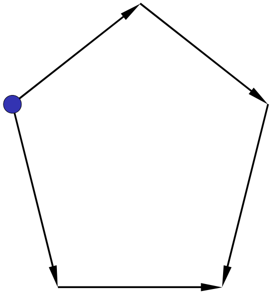

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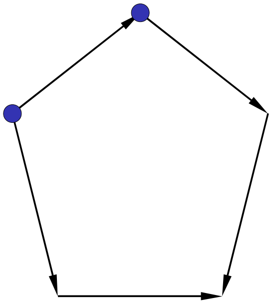

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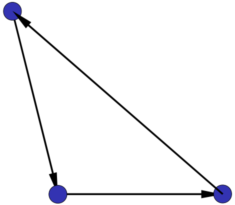

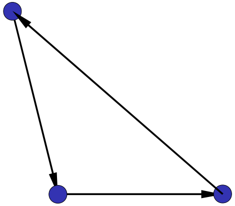

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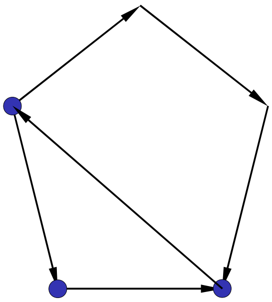

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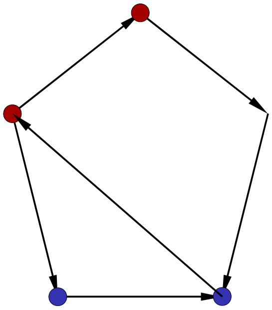

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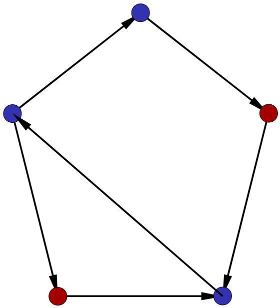

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- and again

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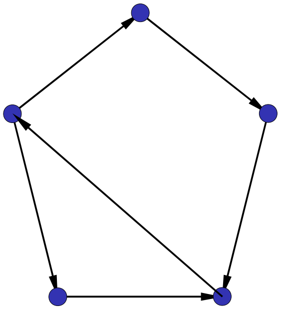

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- lets take only the elements from this smooth digraph
- element from inside is congruent to the element from outside
- and again
- and we obtained a reduction to the inside the congruence block

Part III

In this section we prove the main theorem of this paper. As every cyclic term is a Taylor term Theorem will follow immediately when we prove:

Theorem

Let $\mathbf{A}$ *be a finite algebra in a Taylor variety and let* p *be a prime such that* $p > |A|$*. Then* $\mathbf{A}$ *has a* p*-ary cyclic term operation.*

The remainder of this section is devoted to the proof of this theorem.
We fix a prime number p, we fix a Taylor variety $\mathcal{V}$ and we consider a minimal counterexample to the theorem with respect to the size of A. Thus $\mathbf{A}$ is a finite algebra in $\mathcal{V}$, $p > |A|$, and for all $\mathbf{B} \in \mathcal{V}$ with $|B| < |A|$, $\mathbf{B}$ has a cyclic term of arity p i.e., by Lemma , every cyclic subuniverse of $\mathbf{B}^p$ contains a constant tuple.
An easy reduction proving the following claim can be found in (its proof follows the lines of the proof of Corollary).

Claim

$\mathbf{A}$ is simple.

From Lemma it follows that there exists a cyclic subalgebra $\mathbf{R}$ of $\mathbf{A}^p$ containing no constant tuple. Let $\mathbf{R}_k$, $k = 1, 2, \dots, p$ denote the projection of $\mathbf{R}$ to the first k coordinates, that is

$$R_k = \{(a_0, a_1, \dots, a_{k-1}) : (a_0, \dots, a_{p-1}) \in R\}.$$

Note that, from the cyclicity of R, it follows that for any i we have

$$R_k = \{(a_i, a_{i+1}, \dots, a_{i+k-1}) : (a_0, \dots, a_{p-1}) \in R\},$$

where indices are computed modulo p. In next claim we show that R is subdirect in $\mathbf{A}^p$:

Claim

$R_1 = A$.

Proof.

The projection of R to any coordinate is a subalgebra of $\mathbf{A}$. From cyclicity of R it follows that all projections are equal, say to B. B is a subuniverse of $\mathbf{A}$ and if it is a proper subset of A, then $R \leq_S \mathbf{B}^p$ contains a constant tuple by the minimality assumption, a contradiction. □

We will prove the following two claims by induction on $n = 1, 2, \dots, p$. Note that for $n = 1$ both claims are valid and that property (P1) for $n = p$ contradicts the absence of a constant tuple in R.

(P1) There exists $\mathbf{I} \lhd\lhd \mathbf{A}$ such that $\mathbf{I}^n \lhd\lhd \mathbf{R}_n$.

(P2) If $\mathbf{I}_1, \dots, \mathbf{I}_n \lhd\lhd \mathbf{A}$ and $(I_1 \times \dots \times I_n) \cap R_n \neq \emptyset$, then $\mathbf{I}_1 \times \dots \times \mathbf{I}_n \lhd\lhd R_n$.

We assume that (P1) and (P2) hold for some $n \in \{1, \dots, p-1\}$ and we aim to prove these properties for $n+1$. We fix $\mathbf{I} \lhd\lhd \mathbf{A}$ such that $\mathbf{I}^n \lhd\lhd \mathbf{R}_n$ guaranteed by (P1). Let

$$S = \{((a_0, \dots, a_{n-1}), a_n) : (a_0, \dots, a_n) \in R_{n+1}\}$$

and let $\mathbf{S}$ denote the subalgebra of $\mathbf{A}^{n+1}$ with universe S. Thus $\mathbf{S}$ is basically $\mathbf{R}_{n+1}$, but we look at it as a (subdirect) product of two algebras $\mathbf{R}_n$ and $\mathbf{A}$: $S \leq_S \mathbf{R}_n \times \mathbf{A}$.
The aim of next few claims is to show that S is linked. First we show, that it is enough to have a "fork".

Claim

If there exist $\mathbf{a} \in \mathbf{R}_n$ and $b, b' \in A$, $b \neq b'$ such that $(\mathbf{a}, b), (\mathbf{a}, b') \in S$, then S is linked.

Proof.

Let $k = |A|$. We define a binary relation $\sim$ on A by putting $b \sim b'$ if and only if there exist tuples $\mathbf{a}^1, \dots, \mathbf{a}^k \in R_n$ and elements $b = c_0, c_1, \dots, c_k = b' \in A$ such that for every $i \in \{1, 2, \dots, k\}$ we have

$$(\mathbf{a}^i, c_{i-1}), (\mathbf{a}^i, c_i) \in S.$$

It is clearly reflexive and symmetric. It is also transitive as we have chosen k big enough. It follows immediately form the definition that $\sim$ is a subuniverse of $\mathbf{A}^2$.
Therefore $\sim$ is a congruence of $\mathbf{A}$. Moreover, from the assumption of the claim it follows that $\sim$ is not the smallest congruence (as $b \sim b'$ for $b \neq b'$). Since $\mathbf{A}$ is simple, $\sim$ is the full relation on A and therefore S is linked. □

The next claim shows that S is linked in case that $\mathbf{A}$ has no proper absorbing subuniverses.

Claim

If $I = A$ then S is linked.

Proof.

From (P1) we have $R_n = A^n$. If there are $(a_0, \dots, a_{p-1}), (b_0, \dots, b_{p-1}) \in R$ such that $a_i \neq b_i$ for some i and $a_0 = b_0$, $a_1 = b_1$, $\dots$, $a_{n-1} = b_{n-1}$, then, by cyclically shifting these tuples, we obtain $a'_0 = b'_0$, $\dots$, $a'_{n-1} = b'_{n-1}$, but $a_n \neq b_n$ — a fork and Claim proves that S is linked.
In the other case, tuples in R are determined by the first n projections, thus $|R| = |R_n| = |A|^n$. Consider the mapping $\sigma : R \to R$ sending a tuple $(a_0, \dots, a_{p-1}) \in R$ to its cyclic shift $(a_1, \dots, a_{p-1}, a_0) \in R$. Clearly, σ is a permutation of R satisfying $\sigma^p = \mathrm{id}$. Now p is a prime number and $|R| = |A|^n$ is not divisible by p (as $p > |A|$), therefore σ has a fixed point — a constant tuple, a contradiction. □

The harder case is when $I \neq A$. We need two more auxiliary claims.

Claim

If $I \neq A$ then there exists $\mathbf{J} \lhd\lhd \mathbf{A}$ such that $I \neq J$ and $(I^n \times J) \cap R_{n+1} \neq \emptyset$.

Proof.

Observe that $I^p \cap R$ is a cyclic subuniverse of $\mathbf{I}^p$ without a constant tuple. Therefore $I^p \cap R$ is empty. On the other hand $I^n \cap R \neq \emptyset$ by (P1), so that there exists a greatest number k, $n \leq k < p$, such that $(I^k \times A^{p-k}) \cap R$ is nonempty. Consider the set

$$X = \{a : (a_0, \dots, a_{k-1}, a) \in R_{k+1}, \ \ a_0, \dots, a_{k-1} \in I\}.$$

It is easy to check that X is an absorbing subuniverse of $\mathbf{A}$. As $I^{k+1} \cap R$ is empty, X is disjoint from I. Let J be a minimal absorbing subuniverse of $\mathbf{X}$. We have $J \lhd\lhd \mathbf{A}$ (as $J \lhd\lhd \mathbf{X} \lhd \mathbf{A}$), $I \neq J$ and $(I^k \times J) \cap R_{k+1} \neq \emptyset$. We take a tuple in R whose projection to the first $(k+1)$ coordinates lies in $I^k \times J$, and shift it $(k-n)$ times to the left (recall that $k - n \geq 0$). This tuple shows that $(I^n \times J) \cap R_{n+1}$ is nonempty. □

Similarly we can show that there exists a minimal absorbing subalgebra $\mathbf{J}'$ of $\mathbf{A}$ distinct from I such that $(J' \times I^n) \cap R_{n+1}$ is nonempty.
We consider the following two subsets of $A \times A$.

$$F = \{(a, b) : \exists \ (a, c_1, \dots, c_{n-1}, b) \in R_{n+1}\}$$

$$E = \{(a, b) : \exists \ (a, c_1, \dots, c_{n-1}, b) \in R_{n+1} \ \& \ c_i \in I \ \forall i\}$$

Let V_1 and V_2 denote the projections of E to the first and the second coordinate, so that $E \subseteq_S V_1 \times V_2$.

Claim

E is a linked subuniverse of $\mathbf{A}^2$ and $V_1, V_2 \lhd \mathbf{A}$.

Proof.

It is straightforward to check that E and F are subuniverses of $\mathbf{A}^2$, that $\mathbf{E} \lhd \mathbf{F}$ and that $\mathbf{V}_1, \mathbf{V}_2 \lhd \mathbf{A}$, where $\mathbf{E}, \mathbf{F}$ denote the subalgebras of $\mathbf{A}^2$ with universes E, F and $\mathbf{V}_1, \mathbf{V}_2$ denote the subalgebras of $\mathbf{A}$ with universes V_1, V_2. From Claim we know that $F \leq_S A \times A$.
Similarly as in the proof of Claim we will show that F is linked. Let $k = |A|$ and let us define a congruence $\sim$ on $\mathbf{A}$ by putting $b \sim b'$ if and only if there are $a_1, a_2, \dots, a_k, b = b_0, b_1, \dots, b_k = b' \in A$ such that for all $i \in \{1, 2, \dots, k\}$

$$(a_i, b_{i-1}), (a_i, b_i) \in F.$$

The proof that $\sim$ is a congruence follows exactly as in Claim .
Take an arbitrary tuple $(a_0, \dots, a_{p-1}) \in R$. As p is a prime greater than $|A|$ we can find indices i, j such that $a_i = a_j$ and $a_{i+n} \neq a_{j+n}$ (indices computed modulo p) as otherwise there would be a constant tuple in R. The pairs (a_i, a_{i+n}) and (a_j, a_{j+n}) are in F (by shifting $(a_0, \dots, a_{p-1})$), therefore $\sim$ is not the smallest congruence. Since $\mathbf{A}$ is simple, $\sim$ is the full congruence on $\mathbf{A}$, thus F is linked. By Corollary .(i), E is linked as well. □

Now we can finally show that S is linked.

Claim

S is linked.

Proof.

From Claim and the remark following it we know that $(a, b'), (a', b) \in E$ for some $a, b \in I$, $a' \in J'$, $b' \in J$, $J, J' \lhd\lhd \mathbf{A}$, $I \neq J$, $I \neq J'$. As E is linked, we can find elements $a = c_0, c_1, \dots, c_{2i} = a'$ such that $c_0, c_2, \dots, c_{2i} \in V_1$, $c_1, c_3, \dots, c_{2i-1} \in V_2$ and $(c_{2j}, c_{2j+1}), (c_{2j+2}, c_{2j+1}) \in E$ for all $j = 0, 1, \dots, i-1$. By Corollary .(v) (used for $\mathbf{E} \leq_S \mathbf{V}_1 \times \mathbf{V}_2$) we can assume that all the elements $c_0, \dots c_{2i}$ lie in minimal absorbing subuniverses of $\mathbf{V}_1$ or $\mathbf{V}_2$ (which are also minimal absorbing subuniverses of $\mathbf{A}$, since $V_1, V_2 \lhd \mathbf{A}$). It follows that there exist $w \in W \lhd\lhd \mathbf{V}_1$ and $u \in U \lhd\lhd \mathbf{V}_2$, $v \in V \lhd\lhd \mathbf{V}_2$ such that $(w, u), (w, v) \in E$, $U \neq V$. Therefore there exist $a_1, \dots, a_{n-1}, a'_1, \dots, a'_{n-1} \in I$ such that $(w, a_1, \dots, a_{n-1}, u), (w, a'_1, \dots, a'_{n-1}, v) \in R_{n+1}$.
From the induction hypotheses (P2) we know that $W \times I^{n-1} \lhd\lhd R_n$. Also $V \lhd\lhd \mathbf{A}$ and $((W \times I^{n-1}) \times V) \cap S) \neq \emptyset$. By Corollary .(ii), $((W \times I^{n-1}) \times V) \cap T \leq_S (W \times I^{n-1}) \times V$. In particular, there exists $v' \in V$ such that $(w, a_1, \dots, a_{n-1}, v') \in R_{n+1}$. Now recall that $(w, a_1, \dots, a_{n-1}, u) \in R_{n+1}$ and observe that u and v' are distinct, since they lie in different minimal absorbing subuniverses. Then S is linked by Claim . □

We are ready to prove (P2) for $n+1$.

Claim

(P2) holds for $n+1$.

Proof.

Let $\mathbf{I}_1, \dots, \mathbf{I}_{n+1}$ be absorbing subalgebras of $\mathbf{A}$ such that $(I_1 \times \dots \times I_{n+1}) \cap R_{n+1} \neq \emptyset$. Now S is a linked subdirect subuniverse of $\mathbf{R}_n \times \mathbf{A}$, $I_1 \times \dots \times I_n$ is a minimal absorbing subuniverse of $\mathbf{R}_n$ (from the induction hypotheses (P2)), $I_{n+1} \lhd\lhd \mathbf{A}$ and $((I_1 \times \dots \times I_n) \times I_{n+1}) \cap S \neq \emptyset$. By Corollary .(iii), $(I_1 \times \dots \times I_n) \times I_{n+1}$ is a minimal absorbing subuniverse of $\mathbf{S}$ and thus $I_1 \times \dots \times I_{n+1}$ is a minimal absorbing subuniverse of $\mathbf{R}_{n+1}$. □

To prove (P1) for $n+1$ we define a digraph on R_n by putting

$$((a_0, \dots, a_{n-1}), (a_1, \dots, a_n)) \in E$$

whenever $(a_0, \dots, a_n) \in R_{n+1}$. We want to apply Theorem to obtain a loop in the digraph $\mathbb{G} = (R_n, E)$ in a minimal absorbing subuniverse of $\mathbf{R}_n$.
Observe that E is a subuniverse of $\mathbf{R}_n^2$. Next we show that I^n is contained in a weak component of $\mathbb{G}$.

Claim

Any two elements of I^n are in the same weak component of the digraph $\mathbb{G}$.

Proof.

The set $X = \{x : (a_0, \dots, a_{n-1}, x) \in R_{n+1}, \ \ a_0, \dots, a_{n-1} \in I\}$ is an absorbing subuniverse of $\mathbf{A}$. Let X_0 be a minimal absorbing subuniverse of the algebra $\mathbf{X}$ with universe X. We have found $X_0 \lhd \mathbf{A}$ such that $(I^n \times X_0) \cap R_{n+1} \neq \emptyset$. Similarly we can find $X_1, X_2, \dots, X_{n-1}$ such that $(I^{n-i} \times X_0 \times X_1 \times \dots \times X_i) \cap R_{n+1} \neq \emptyset$ for all $i = 0, 1, \dots, n-1$. From (P2) for $n+1$ (Claim) it follows that $I^{n-i} \times X_0 \times X_1 \times \dots \times X_i \subseteq R_{n+1}$ for all i. Now choose arbitrary elements $x_i \in X_i$ and take any tuple $(b_0, \dots, b_{n-1}) \in I^n$. Since, for all $i = 0, 1, \dots, n-1$, the tuple $(b_i, \dots, b_{n-1}, x_0, x_1, \dots, x_i)$ belongs to R_{n+1}, the vertices $(b_i, \dots, b_{n-1}, x_0, \dots, x_{i-1})$ and $(b_{i+1}, \dots, b_{n-1}, x_0, \dots, x_i)$ are in the same weak component of $\mathbb{G}$. Therefore the vertex $(b_0, \dots, b_{n-1})$, which was an arbitrarily chosen vertex in I^n, is in the same weak component as the vertex $(x_0, \dots, x_{n-1})$. □

The last assumption of Theorem is proved in the next claim.

Claim

The weak component of $\mathbb{G}$ containing I^n has algebraic length 1.

Proof.

As E is connected, we reason as in the proof of Claim 7, to get $b = c_0, c_1, \dots, c_{2i}, b$ in minimal absorbing subuniverses of $\mathbf{A}$ such that $b \in I$, and $(c_{2j}, c_{2j+1}), (c_{2j+2}, c_{2j+1}) \in E$ for all $j = 0, \dots, i-1$ and $(c_{2i}, b) \in E$. Property (P2) for $n+1$ (Claim) proves that $(c_{2j}, b, \dots, b, c_{2j+1}), (c_{2j+2}, b, \dots, b, c_{2j+1}) \in R_{n+1}$ for all $j = 0, \dots, i-1$ and $(c_{2i}, b, \dots, b, b) \in R_{n+1}$. Which gives rise to a $(1, j)$-fence connecting, in $\mathbb{G}$, the tuple $(b, \dots, b)$ to the tuple $(c_{2i}, b, \dots, b)$ (as $c_0 = b$). As $((c_{2i}, b, \dots, b), (b, \dots, b)) \in E$ we showed that the algebraic length of the weak component containing I^n is one. □

By Theorem there exists a loop inside a minimal absorbing subuniverse K of $\mathbf{R}_n$. Since the projection J of K to the first coordinate is a minimal absorbing subuniverse of $\mathbf{A}$, we actually get an element $a \in J \lhd\lhd \mathbf{A}$ such that $(a, \dots, a) \in R_{n+1}$. Now (P1) follows from (P2) and the proof of Theorem is concluded.

Thank you