Nilpotents in a semigroup of partial automorphisms of a rooted tree

Eugenia Kochubinska

Kiev National Taras Shevchenko University

22 June 2010

Outline

- Partial automorphisms semigroup of rooted trees and partial wreath product.

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- References.

Let T be a rooted k-level n-regular tree.
Let PAut T be the semigroup of partial automorphisms of the tree T, defined on a connected subgraph containing root and preserving the level of vertices.
PAut T is an inverse semigroup.

For a semigroup S define S^{PX} by

$$S^{PX} = \{f : \mathrm{dom}(f) \subseteq X \to S\}.$$

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For $f, g \in S^{PX}$, define the product fg by:

$$(fg)(x) = f(x)g(x), \quad x \in \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g).$$

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If $a \in \mathcal{IS}(X), f \in S^{PX}$, we define f^a by:

$$(f^a)(x) = f(x^a), \quad x \in \mathrm{dom}(f^a) = \mathrm{dom}(a) \cap \{x : x^a \in \mathrm{dom}(f)\}$$

Definition

Partial wreath product of semigroup S with semigroup (P, X) of partial transformations of the set X is the set

$$\{(f, a) \in S^{PX} \times (P, X) \mid \operatorname{dom}(f) = \operatorname{dom}(a)\}$$

with composition defined by $(f, a) \cdot (g, b) = (fg^a, ab)$.

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The partial wreath product of two semigroup is a semigroup itself. Moreover, partial wreath product of inverse semigroups is an inverse semigroup.

Proposition

Let T be a rooted k-level n-regular tree. Then

$$\mathrm{PAut}\, T \cong \mathcal{IS}_n \wr_p \ldots \wr_p \mathcal{IS}_n = \wr_p^k \mathcal{IS}_n.$$

In a paper [1] it has been shown that there about 1.8 million of non-equivalent semigroups of order 8 and 99% of them are nilpotent.

Let S be a semigroup with a zero element 0.
An element $a \in S$ is called *nilpotent* if for some $k \in \mathbb{N}$ $a^k = 0$
Semigroup S is called *nilpotent* if for some $k \in \mathbb{N}$

$$a_1 \cdot a_2 \cdot \ldots \cdot a_k = 0$$

for arbitrary $a_1, a_2, \ldots, a_k \in S$.

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First class includes nilpotent subsemigroup containing semigroup zero 0.

In this case we have $T^k = \{0\}$ for some $k > 0$.

Second class includes subsemigroups of S, which are nilpotent as semigroups, but their zero element differs from 0.

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Second class includes subsemigroups of S, which are nilpotent as semigroups, but their zero element differs from 0.
In this case we have $T^k = \{e\}$ for some idempotent $e \neq \mathrm{id}$ and some $k \geq 1$.
We will call these subsemigroups *proper nilpotent subsemigroups*.

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We will call these subsemigroups *proper nilpotent subsemigroups*.
A nilpotent subsemigroup $T \subset S$ is called *maximal nilpotent subsemigroup*, if it is not contained in any other nilpotent subsemigroup $T' \subset S$, $T \neq T'$.

For a semigroup $\mathcal{IS}_n$ there is a one-to-one correspondence between maximal nilpotent subsemigroups and partitions of the set $\{1, 2, \ldots, n\}$ into disjoint union of ordered blocks:

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If e is an idempotent, which is the zero in a $T \subset \mathcal{IS}_n$, and $\mathrm{dom}(e) = \{a_1, a_2, \ldots a_k\}$, then every maximal nilpotent subsemigroup corresponds to a permutation $b_1, b_2, \ldots, b_k$ of $a_1, a_2, \ldots, a_k$ and has the form:

$$T = \Big\{\sigma \in \mathcal{IS}_n \,|\, \mathrm{dom}(e) \subseteq \mathrm{dom}(\sigma); \sigma(x) = x \text{ for all } x \in \mathrm{dom}(e);$$
$$\sigma(b_i) = b_j \text{ implies } i < j \text{ for all } b_i \notin \mathrm{dom}(e)\Big\}$$

Details can be found in [2].

Consider maximal nilpotent subsemigroups (those containing the semigroup zero) in a slightly more general setting.
Let P be an inverse semigroup.
An element $(f, a) \in P \wr_p \mathcal{IS}_n$ is nilpotent iff $a \in \mathcal{IS}_n$ is nilpotent.

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Proposition

Let S be a maximal nilpotent subsemigroup of the semigroup $\mathcal{IS}_n$. Then subsemigroup $P \wr_p S$ is a maximal nilpotent subsemigroup of the semigroup $P \wr_p \mathcal{IS}_n$. Moreover, every maximal nilpotent subsemigroup of semigroup $P \wr_p \mathcal{IS}_n$ is of this form.

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Corollary

Maximal nilpotent subsemigroup of semigroup $\wr_p^k \mathcal{IS}_n$ are those having the form $(\wr_p^{k-1} \mathcal{IS}_n) \wr_p S$, where S is a maximal nilpotent subsemigroup of the semigroup $\mathcal{IS}_n$.

Now consider proper maximal nilpotent subsemigroups.
Let T be a proper nilpotent subsemigroup of $\mathrm{PAut}\, T$ with a zero $e \in E(\mathrm{PAut}\, T)$. Denote by T_x the maximal subtree of T such that its root is $x \in VT$ and none of the edge of T_x is in $\mathrm{dom}(e)$.

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Theorem

Proper maximal subsemigroup of $\mathrm{PAut}\, T$ *is (canonically) isomorphic to*

$$\prod_{x \in \mathrm{dom}(e)} Nilp_x,$$

where $Nilp_x$ *is a maximal nilpotent subsemigroup of* $\mathrm{PAut}\, T_x$.

The cardinality of a maximal subsemigroup of $P \wr_p \mathcal{IS}_n$ is equal to

$$|P|^n B_n\left(\frac{1}{|P|}\right),$$

where $B_n(x)$ denotes the nth Bell polynomial.
Consequently, the cardinality of a maximal subsemigroup of $\wr_p^k \mathcal{IS}_n$ is

$$\left|\wr_p^{k-1} \mathcal{IS}_n\right|^n B_n\left(\frac{1}{\left|\wr_p^{k-1} \mathcal{IS}_n\right|}\right).$$

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Denote $F^k(x) = \underbrace{F(F \dots (F}_{k}(x)) \dots)$.

$$|\wr_p^k \mathcal{IS}_n| = F^k(1),$$

where $F(x) = \sum_{i=1}^n \binom{n}{i}^2 i! x^i$

The number of proper maximal nilpotent subsemigroups of PAut T with a zero $e \in E(\text{PAut } T)$ equals

$$\prod_{x \in \text{dom}(e)} (k_x)!,$$

where k_x is the number of vertices of the first level of the tree T_x.

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