Maltsev digraphs have a majority polymorphism

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Basic definitions

- A digraph will be a directed graph with loops allowed, i.e. the relational structure $G = (V, E)$ with $E \subseteq V^2$.
- Given a graph, we can define the algebra of its idempotent polymorphisms $\text{Pol} \ G$.
- A polymorphism $m : V^3 \to V$ is Maltsev if for all $x, y \in V$ we have
  
  $$m(x, y, y) = x \quad m(x, x, y) = y.$$ 

- A polymorphism $M : V^3 \to V$ is a majority if for all $x, y \in V$ we have
  
  $$M(y, x, x) = M(x, y, x) = M(y, x, x) = x.$$
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Maltsev ⇒ majority

- We will call a digraph $G$ Maltsev resp. having a majority if $\text{Pol} \ G$ contains a Maltsev resp. majority polymorphism.
- In general algebras, having Maltsev operation does not imply having majority (consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$).
- However, we show that if a digraph is Maltsev then it does have a majority.
- From now on we will assume that $G$ is has a Maltsev operation $m$ and is smooth.
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Rectangularity

- Let $x, y, x', y'$ be vertices of $G$ and let $(x, y), (x', y'), (x', y) \in E$.
- Now apply the Maltsev polymorphism $m$ and we get . . .
- . . . that $(x, y') \in E$ as well.

- We say that $E$ is rectangular.
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![Diagram](image)

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Rectangularity

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R$^+$ and R$^-$

- For $\nu$ in $V$, we will denote by $\nu^+$ the vertex set $
\{ u \in V(G) : (\nu, u) \in E(G) \}$ by $\nu^-$ the vertex set
$\{ u \in V(G) : (u, \nu) \in E(G) \}$.
- For $u$, $\nu$ vertices of $G$, we write $uR^+ \nu$ if $u^+ = \nu^+$ and $uR^- \nu$ if $u^- = \nu^-$. 

• In the picture, we have $x^+ = y^+$, therefore $xR^+ y$. 

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R+ and R−

- For ν in V, we will denote by ν+ the vertex set \( \{ u \in V(G) : (ν, u) \in E(G) \} \) by ν− the vertex set \( \{ u \in V(G) : (u, ν) \in E(G) \} \).

- For u, ν vertices of G, we write \( uR^+ν \) if \( u^+ = ν^+ \) and \( uR^-ν \) if \( u^- = ν^- \).

- In the picture, we have \( x^+ = y^+ \), therefore \( xR^+y \).
For $v$ in $V$, we will denote by $v^+$ the vertex set 
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by $v^-$ the vertex set 
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\[
\begin{array}{c}
\begin{tikzpicture}
\node (x) at (0,0) [circle, fill=black] {$x$};
\node (y) at (0,-1) [circle, fill=black] {$y$};
\draw[->] (x) to (y);
\end{tikzpicture}
\end{array}
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\begin{cases}
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$R^+$ and $R^-$ are nice

- As $E$ is rectangular, we obtain the following:
- The relations $R^+$ and $R^-$ are equivalences on $V$.
- The mapping $\phi : E \mapsto E^+$ is a bijection from the set of equivalence classes of $R^+$ to the set of equivalence classes of $R^-$. 
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\[ R^+ \text{ and } R^- \text{ in a picture} \]

Graph $G$

$E \subseteq V^2$

$R^+$-classes  \quad  $R^-$-classes
The graphs $G^+$ and $G^-$

- Given $G$, we define the graph $G^+$ whose vertices are the equivalence classes of $R^+$ and $(U, V) \in E(G^+)$ iff there exist vertices $u \in U, v \in V$ with $(u, v) \in E(G)$.
- We define $G^-$ similarly.
- A little thought gives us that $G^+$ and $G^-$ are isomorphic.
- It turns out that if $G$ is Maltsev then so is $G^+$.
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Proof by induction

We are now ready for a proof by induction.

Assume that $G$ is the smallest Maltsev graph without a majority operation.

If $|V(G^+)| = |V(G)|$ then $G$ is a graph of a permutation and we win.

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Extending the majority

- Else, we have a majority operation $M^+$ on $G^+$ and $M^-$ on $G^-$ which we can extend to $M$ on $G$ by demanding that

$$[M(x, y, z)]_{R^+} = M^+([x]_{R^+}, [y]_{R^+}, [z]_{R^+})$$
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- Examining $R^+$ and $R^-$, we discover that such an $M$ always exists and is a majority polymorphism of $G$. 
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CSP complexity

- If $G$ is a graph, add constants (names of vertices) to the language of $G$ and consider the problem $\text{CSP}(G_c)$.
- If $G$ is Maltsev then we already know that $\text{CSP}(G_c)$ is in P...
- ...however, if $G$ has both Maltsev and majority then $\text{CSP}(G_c)$ is even easier: solvable in deterministic logarithmic space (a result by V. Dalmau and B. Larose).
- Therefore we have $G$ Maltsev $\implies \text{CSP}(G_c)$ is solvable in logspace.
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Open problems

- Is it possible to generalize the result to the case when $G$ has several edge relations?
- What other implications of the type “$G$ has $t$ then $G$ has $s$” hold in graphs but not for general algebras?
- Maybe some such implications hold for all finitely presented algebras?
- It would also be interesting to estimate the number of Maltsev graphs on $n$ vertices.
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Thanks for your attention.