Maltsev digraphs have a majority polymorphism

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Outline

Basic definitions

- A digraph will be a directed graph with loops allowed, i.e. the relational structure $G = (V, E)$ with $E \subset V^2$.
- Given a graph, we can define the algebra of its idempotent polymorphisms Pol G.
- A polymorphism $m : V^3 \to V$ is Maltsev if for all $x, y \in V$ we have

$$m(x, y, y) = x \quad m(x, x, y) = y.$$

- A polymorphism $M : V^3 \to V$ is a majority if for all $x, y \in V$ we have

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Maltsev $\Rightarrow$ majority

- We will call a digraph G Maltsev resp. having a majority if Pol G contains a Maltsev resp. majority polymorphism.
- In general algebras, having Maltsev operation does not imply having majority (consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$).
- However, we show that if a digraph is Maltsev then it does have a majority.
- From now on we will assume that G is has a Maltsev operation m and is smooth.

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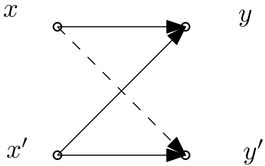
Rectangularity

- Let x, y, x', y' be vertices of G and let $(x, y), (x', y'), (x', y) \in E$.
- Now apply the Maltsev polymorphism m and we get ...
- ... that $(x, y') \in E$ as well.



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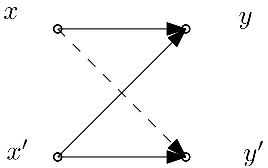

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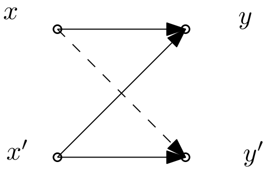

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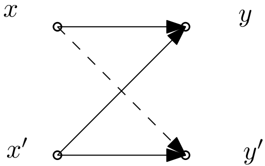

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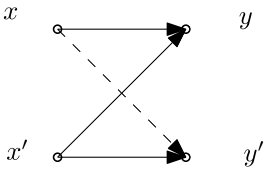

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R^+ and R^-

- For v in V, we will denote by v^+ the vertex set $\{u \in V(G) : (v, u) \in E(G)\}$ by v^- the vertex set $\{u \in V(G) : (u, v) \in E(G)\}$.
- For u, v vertices of G, we write uR^+v if $u^+ = v^+$ and uR^-v if $u^- = v^-$.
- In the picture, we have $x^+ = y^+$, therefore xR^+y.

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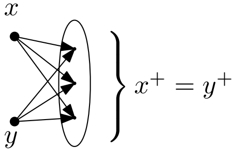

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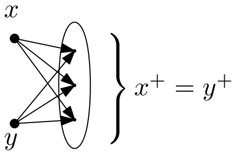

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R^+ and R^- are nice

- As E is rectangular, we obtain the following:
- The relations R^+ and R^- are equivalences on V.
- The mapping $\phi : E \mapsto E^+$ is a bijection from the set of equivalence classes of R^+ to the set of equivalence classes of R^-.

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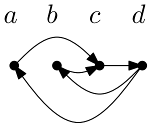
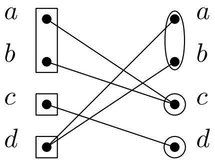
R^+ and R^- in a picture



Graph G

$E \subseteq V^2$



R^+-classes R^--classes

The graphs G^+ and G^-

- Given G, we define the graph G^+ whose vertices are the equivalence classes of R^+ and $(U, V) \in E(G^+)$ iff there exist vertices $u \in U, v \in V$ with $(u, v) \in E(G)$.
- We define G^- similarly.
- A little thought gives us that G^+ and G^- are isomorphic.
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Proof by induction

- We are now ready for a proof by induction.
- Assume that G is the smallest Maltsev graph without a majority operation.
- If $|V(G^+)| = |V(G)|$ then G is a graph of a permutation and we win.
- Else. . .

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Extending the majority

- Else, we have a majority operation M^+ on G^+ and M^- on G^- which we can extend to M on G by demanding that

$$[M(x,y,z)]_{R^+} = M^+([x]_{R^+},[y]_{R^+},[z]_{R^+})$$
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CSP complexity

- If G is a graph, add constants (=names of vertices) to the language of G and consider the problem CSP(G_c).
- If G is Maltsev then we already know that CSP(G_c) is in P...
- ...however, if G has both Maltsev and majority then CSP(G_c) is even easier: solvable in deterministic logarithmic space (a result by V. Dalmau and B. Larose).
- Therefore we have G Maltsev $\Rightarrow$ CSP(G_c) is solvable in logspace.

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Open problems

- Is it possible to generalize the result to the case when G has several edge relations?
- What other implications of the type "G has t then G has s" hold in graphs but not for general algebras?
- Maybe some such implications hold for all finitely presented algebras?
- It would also be interesting to estimate the number of Maltsev graphs on n vertices.

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