Serre's Problem and p-Schreier Varieties of Semimodules

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1. Preliminaries

By the classical Nielsen-Schreier theorem (see, for example, [11]), every subgroup of a free group is itself free. This fact stimulated a strong interest in establishing the analogs of the Nielsen-Schreier theorem in different varieties of algebras (for some activity in this direction one may consult, for instance, [3]). As a result, *Schreier varieties* of algebras, namely varieties in which any subalgebra of any free algebra is itself free, emerged. Varieties of all groups, all abelian groups, all abelian groups of prime exponent, varieties of all Lie algebras over fields, as well as varieties of all nonassociative algebras over fields are all examples of Schreier varieties. However, in homological algebra, *projective algebras* — algebras which are retracts of free algebras — play a very special and important role. Therefore, combining the concepts of free and projective algebra, one naturally comes up with the concept of a *p-Schreier variety* — a variety whose projective algebras are all free. Perhaps the most well-known results concerning p-Schreier varieties are the following: the category of modules over a local ring is a p-Schreier variety, i.e., the homological result that every projective module over a local ring is free, (see, for example, Corollary 26.7 in [1] , or Theorem 19.29 in [9]), originally proved by Kaplansky; and a confirmation of Serre's famous conjecture that the category of modules over a polynomial ring $R[x_1, x_2, \ldots, x_n]$ over a field R is a p-Schreier variety, independently proved by D. Quillen and A. Suslin in 1976.

Remark-Short Historical Notes (Lam T. Y., Serr's Problem on Projective Modules, Springer, 2006):

" "Serre's Conjecture", for the most part of the second half of the 20th century, referred to the famous statement made by J.-P. Serre in 1955, to the effect that one did not know if finitely generated projective modules were free over a polynomial ring $K[x_1, x_2, \ldots, x_n]$, where K is a field. This statement was motivated by the fact that the affine scheme defined by $K[x_1, x_2, \ldots, x_n]$ is the algebro-geometric analogue of the affine n-space over K. In topology, the n-space is contractible, so there only trivial bundels over it. *Would the analogue of the latter also hold for the n-space in algebraic geometry?* Since algebraic vector bundels over Spec $K[x_1, x_2, \ldots, x_n]$ correspond to finitely generated projective modules over $K[x_1, x_2, \ldots, x_n]$, the question was tantamount to whether such projective modules were free, for any base field K.

It was quite clear that Serre intended his statement as an *open problem* in the sheaf-theoretical framework of algebraic geometry, which was just beginning to emerge in the mid-1950s. Nowhere in his published writings had Serre speculated, one way or another, upon the possible outcome of his problem."

Also, mention the Bass' result on big projectives.

Theorem (Bass' Theorem). Not finitely generated projective modules over Noetherian commutative integral domains are free.

In this talk, we discuss p-Schreier varieties in the context of semimodules over semirings, implicitly studied in [4] , [13] , and [8] .

Recall ([2]) that a *semiring* is an algebra

$(R, +, \cdot, 0, 1)$ such that the following conditions are satisfied:

(1) $(R, +, 0)$ is a commutative monoid with identity element 0;

(2) $(R, \cdot, 1)$ is a monoid with identity element 1;

(3) Multiplication is distributive over addition from both sides;

(4) $0r = 0 = r0$ for all $r \in R$.

A semiring R, which is not a ring, very often is called a *proper semiring*; and a semiring R is *zerosumfree* if the following is true: $\forall a, b \in R(a + b = 0 \Rightarrow a = 0 \;\&\; b = 0)$.

As usual, a *left* R-*semimodule* over the semiring R is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(r, m) \mapsto rm$ from $R \times M$ to M which satisfies the following identities for all $r, r' \in R$ and $m, m' \in M$:

(1) $(rr')m = r(r'm)$;

(2) $r(m + m') = rm + rm'$;

(3) $(r + r')m = rm + r'm$;

(4) $1m = m$;

(5) $r0_M = 0_M = 0m$.

Right semimodules over R and homomorphisms between semimodules are defined in the standard manner. From now on, let $\mathcal{M}$ be the variety of commutative monoids, $\mathcal{M}_R$ and ${}_R\mathcal{M}$ denote the categories of right and left semimodules with the classes of objects $|\mathcal{M}_R|$ and $|{}_R\mathcal{M}|$, respectively, over a semiring R. As usual (see, for example, Chapter 17 of [2]), if R is a semiring, then in the category ${}_R\mathcal{M}$, a free (left) semimodule $\sum_{i\in I} R_i, R_i \cong {}_R R$, $i \in I$, with a basis set I is a direct sum (a coproduct) of I copies of ${}_R R$. A semimodule ${}_R P \in |{}_R\mathcal{M}|$ is *projective* if for any surjective homomorphism $f : M \longrightarrow N$ between left semimodules $M, N \in |{}_R\mathcal{M}|$, and any R-homomorphism $g : P \longrightarrow N$, there exists an R-homomorphism $h : P \longrightarrow M$ such that $f \circ h = g$. Of course, we are free to use the obvious right-sided analogs of the notions we have just introduced.

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2. Division semirings over which all semimodules are projec

A subsemimodule $\alpha(M)$ of a semimodule $M \in |\mathcal{M}_R|$ is called a *projection* of M (onto $\alpha(M)$) if $\alpha(M)$ is the image of an idempotent endomorphism

$\alpha : M \to M$; and one has the following general, almost obvious, observation in a semimodule context.

Proposition 2.1. A semimodule $P_R \in |\mathcal{M}_R|$ is projective iff it is isomorphic to a projection of some free semimodule $F_R \in |\mathcal{M}_R|$. □

From this observation readily follows

Corollary 2.2. For any multiplicatively idempotent element $e \in R$ of a semiring R, the semimodule $(eR)_R$ is projective. □

It is well known (see, for instance, Statement 2.1 in [14]) that division rings are the only rings over which all modules are free, and therefore, categories of all modules over division rings obviously are p-Schreier varieties. In contrast to division rings, we show that all semimodules over a division semiring are projective iff the semiring, in fact, is a division ring.

Theorem 2.3. For a division semiring R the following conditions are equivalent:

(1) All (right, or left) R-semimodules are free.

(2) All (right, or left) R-semimodules are projective.

(3) R is a division ring. □

3. p-Schreier varieties of semimodules over additively π-reg

First consider a variety of semimodules over an additively idempotent semiring R. In this case, it is clear that the additive reduct $(R, +, 0)$, i.e., the commutative monoid $(R, +, 0)$, of the semiring R is, in fact, a semilattice with zero 0, and we have the following important observation.

Proposition 3.1. A category $\mathcal{M}_R$ of right semimodules over an additively idempotent semiring R is not a p-Schreier variety. □

As usual, for given semirings R and S, an R-S-*semimodule* A—in symbols ${}_R A_S$—is a commutative monoid which is both a left R-semimodule and a right S-semimodule, with always $(ra)s = r(as)$; a *bisemimodule homomorphism* $f : {}_R A_S \rightarrow {}_R B_S$ is a monoid homomorphism $f : A \rightarrow B$ with $r(fa)s = f(ras)$ always, and ${}_R\mathcal{M}_S(A, B)$ denotes the set of all bisemimodule homomorphisms from A to B. Clearly, all R-S-semimodules together with bisemimodule homomorphisms form the category ${}_R\mathcal{M}_S$ of R-S-semimodules. In [8] the construction of and the results on the tensor product bifunctor, originally introduced in [7] , were considered in the bisemimodule setting; and among other facts, in Theorem 3.3 of [8] was shown that for any semirings R, S, T, any R-S-bisemimodule ${}_R B_S \in |{}_R\mathcal{M}_S|$, the functor $\mathcal{M}_S(B, -) : {}_T\mathcal{M}_S \rightarrow {}_T\mathcal{M}_R$ is a right adjoint to the functor $- \otimes {}_R B : {}_T\mathcal{M}_R \rightarrow {}_T\mathcal{M}_S$, i.e., $- \otimes {}_R B \dashv \mathcal{M}_S(B, -)$; and for any T-R-bisemimodule ${}_T A_R \in |{}_T\mathcal{M}_R|$, the functor ${}_T\mathcal{M}(A, -) : {}_T M_S \rightarrow {}_R M_S$ is a right adjoint to the functor $A \otimes_R - : {}_R\mathcal{M}_S \rightarrow {}_T\mathcal{M}_S$, i.e., $A \otimes_R - \dashv {}_T\mathcal{M}(A, -)$.

Furthermore, recall two more functors introduced in [8] . Namely, given any semirings R, S and a semiring homomorphism $\pi : R \longrightarrow S$, any right S-semimodule B_S may be considered as a right R-semimodule by *pullback along* π, that is by defining $b \cdot r = b\pi(r)$ for any $b \in B$, $r \in R$. The resulting R-semimodule is written $\pi^{\#}B$, and it is easy to see that the assignments $B \mapsto \pi^{\#}B$ are obviously raised to the *restriction* functor $\pi^{\#}: \mathcal{M}_S \to \mathcal{M}_R$. The restriction functor $\pi^{\#}$ for left semimodules is similarly defined.

In particular, the restriction functor $\pi^{\#}: {}_S\mathcal{M} \longrightarrow {}_R\mathcal{M}$, applied to the left S-semimodule ${}_SS$, gives the R-S-bisemimodule ${}_RS_S = \pi^{\#}S$. Then, tensoring by $\pi^{\#}S$, one has the *extension* functor $\pi_{\#} := - \otimes_R \pi^{\#}S = - \otimes_R S : \mathcal{M}_R \longrightarrow \mathcal{M}_S$, and by Proposition 4.1 of [8] , we have a natural adjunction $\pi_{\#}: \mathcal{M}_R \rightleftarrows \mathcal{M}_S: \pi^{\#}$, i.e., $\pi_{\#} \dashv \pi^{\#}$. Then, the following observation will prove to be useful.

Proposition 3.2. Let $\pi : R \longrightarrow S$ be a semiring homomorphism with a projective right R-semimodule $\pi^{\#}S \in \mathcal{M}_R$. If the category $\mathcal{M}_S$ of right semimodules over a semiring S is not a p-Schreier variety, then $\mathcal{M}_R$ is not a p-Schreier variety, too.

□

Recall (see, for example, [5]) that a commutative monoid $(S,+,0)$ is called π-*regular* (or an epigroup) if every its element has a power in some subgroup of S. Using Clifford representations of commutative inverse monoids (see, for example, Theorem 3.2.1 of [5]), it is easy to show that the last condition is equivalent to the condition that for any $a \in S$ there exist a natural number n and an element $x \in S$ such that $na + x + na = na$. A semiring R is called *additively* π-*regular* iff its additive reduct $(R,+,0)$ is a π-regular monoid. Then, it is clear that a semiring R is additively π-regular iff for any $a \in R$ there exist a natural number n and an element $x \in S$ such that $na + x + na = na$. Note that the element x in the last equation can be chosen to be mutually inverse with the element na, i.e., $na + x + na = na$ and $x + na + x = x$. Indeed, if $na + x + na = na$ for an element $x \in S$, then one can immediately verify that na and $x + na + x$ are mutually inverse. Moreover, as all our semirings contain a multiplicative identity 1, we can just define an *additively* π-*regular* semiring as a semiring R for which there exist a natural number n and an element $x \in R$ such that n1 and x are mutually inverse, i.e., $n1 + x + n1 = n1$ and $x + n1 + x = x$. Note that the class of additively π-regular semirings is sufficiently abundant — it includes the classes of associative rings, additively regular (particularly, additively idempotent) semirings, finite and locally finite semirings (semirings whose finitely generated subsemirings are finite), for example. Also, the categories of semimodules over additively π-regular semirings are certainly of interest, and some recent homological results about them can be found in [6] . Anyway, our next result concerns the categories of semimodules over additively π-regular semirings, extends Proposition 3.1 to the categories of semimodules over additively π-regular semirings, and solves Problem 1 of [8] .

Theorem 3.3. A category $\mathcal{M}_R$ of right semimodules over a proper additively π-regular semiring R is not a p-Schreier variety. □

Corollary 3.4. A category $\mathcal{M}_R$ of right semimodules over an additively π-regular semiring R is a p-Schreier variety iff $\mathcal{M}_R$ is actually a p-Schreier variety of right modules over a ring R. □

The next result straightforwardly follows from Corollary 3.4 and, in fact, extends Theorem 5.7 of [8] to polynomial semirings over additively π-regular semirings.

Corollary 3.5. The categories of right (left) semimodules over the polynomial semirings $R[x_1, x_2, \ldots, x_n]$ over proper additively π-regular semirings R are not p-Schreier varieties. □

4. Varieties of semimodules over cancellative division semi

Following [2] , a semiring R is called *cancellative* if its additive reduct $(R, +, 0)$ is a cancellative monoid, i.e., $a + b = a + c \Rightarrow b = c$ for any a, b and c in R.

Theorem 4.1. *The categories of semimodules over cancellative division semirings are p-Schreier varieties.* □

We conclude the paper with the following variation of Problem 2 of [8] .

Problem. *Are the categories of semimodules over polynomial semirings* $R[x_1, x_2, \dots, x_n]$ *over cancellative semifields* R p-*Schreier varieties?*

References

Anderson, Fuller (1992) (andful:racom) [1] Anderson, F. W., Fuller, K. R. (1992). Rings and Categories of Modules. Springer-Verlag, New York, 1992

Golan (1999) (golan:sata) [2] Golan, J. S. (1999). Semirings and their Applications. Dordrecht-Boston-London: Kluwer Academic Publishers.

Grätzer (1979) (gratzer:ua) [3] Grätzer, G. (1979). Universal Algebra, 2nd Ed. New York-Berlin: Springer-Verlag.

Grillet (1969) (grillet:ofcs) [4] Grillet, P. A. (1969). On free commutative semigroups. J. Natur. Sci. and Math. 9: 71–78

Grillet (2001) (grillet:cs) [5] Grillet, P. A. (2001). Commutative Semigroups. Dordrecht: Kluwer Academic Publishers.

Il’in (2008) (il’in: otaott) [6] Il’in, S. N. (2008). On the applicability to semirings of two theorems from the theory of rings and modules. Mathematical Notes 83: 492 – 499.

Katsov (1997) (katsov:tpies) [7] Katsov, Y. (1997). Tensor products and injective envelopes of semimodules over additively regular semirings. Algebra Colloquium 4(2): 121–131.

Katsov (2004) (katsov:thhs) [8] Katsov, Y. (2004). Toward homological characterization of semirings: Serre’s conjecture and Bass’s perfectness in a semiring context. Algebra Universalis 52: 197–214.

Lam (2001) (lam: afcinr) [9] Lam, T. Y. (2001). A First Course in Noncommutative Rings. 2nd Ed. New York-Berlin: Springer-Verlag.

Mac Lane (1971) (macl:cwm) [10] Mac Lane, S. (1971). Categories for the Working Mathematician. New York-Berlin: Springer-Verlag.

Rotman (1995) (rotman:aitttog) [11] Rotman, J. J. (1995). An Introduction to the Theory of Groups. Fourth edition. New York-Berlin: Springer-Verlag.

Rowen (1991) (rowen:rt) [12] Rowen, L. H. (1991). Ring Theory. Student Edition. Boston, MA: Academic Press, Inc.

Sokratova (2002) (sokr:osocais) [13] Sokratova, O. (2002).

On semimodules over commutative, additively idempotent semirings. Semigroup Forum 64: 1–11.

Tuganbaev (1998) (tugan:smar) [14] Tuganbaev, A. A. (1998). Semidistributive Modules and Rings. Dordrecht-Boston-London: Kluwer Academic Publishers.