On the equational theory of projection lattices of finite von Neumann factors

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0. Modular (ortho)lattices

We consider *modular lattices* (shortly MLs) - we write $a+b$ for joins and ab for meets - and *modular ortholattices* (shortly MOLs), a subject started by Birkhoff and von Neumann [1]. These have constants 0 and 1 and a unary fundamental operation $x \mapsto x'$ which is an *involution* and, moreover, an *orthocomplementation*

$$x'' = x, \quad x \leq y \Rightarrow y' \leq x'; \quad x \oplus x' = 1,$$

The principal examples are the lattices $L(M)$ of all submodules of some R-module M and the lattices

$$L(V, \Phi) = \{X, X^{\perp} \mid X \in L(V), \dim X < \infty\}, \quad X \mapsto X^{\perp}$$

where (V, Φ) is an inner product space. Considering lattice identities $\forall \overline{x}.\ f(\overline{x}) = g(\overline{x})$, we may assume that $f(\overline{x}) \leq g(\overline{x})$ holds in all lattices. In MOLs it suffices to consider identities of the form $\forall \overline{x}.\ t(\overline{x}) = 0$ - put $t = gf'$.

1. Identities in the atomistic case

Proposition 1.1. *If L is an atomistic ML or MOL, then $L \models \forall \overline{x}.\ f(\overline{x}) = g(\overline{x})$ if and only if $[0, u] \models \forall \overline{x},\ f(\overline{x}) = g(\overline{x})$ for all interval subalgebras $[0, u]$ where $\dim u$ is at most the number of occurrences of variables in f and g, together.*

Here, in the case of MOLs, $[0, u]$ is endowed with the orthocomplement $x^u = x'u$.

Lemma 1.2. *In an atomistic ML, if $h(\overline{x})$ is a lattice term with unique occurrence of variables, and if $p \leq h(\overline{a})$ for some atom p, then there is a substitution by atoms such that $p \leq h(\overline{q})$.*

Proof. By induction. If $p \leq h_1(\overline{a}_1) + h_2(\overline{a}_2)$ then there are $p_i \leq h_i(\overline{a}_i)$ such that $p \leq p_1 + p_2$ - this is, in essence, the Theorem on Joins in projective spaces. By inductive hypothesis we have $p_i \leq h_i(\overline{q}_i)$ for some substitutions by atoms. $\square$

Proof. Prop.1. In the lattice case, assume $f(\overline{a}) < g(\overline{a})$. Then there is $p \leq g(\overline{a})$, $p \not\leq f(\overline{a})$. Consider $h(\overline{z})$ with unique occurrence and substitution $h(\overline{a}) = g(\overline{a})$. Then by the Lemma there is a substitution by atoms such that $p \leq h(\overline{q})$. Collect all q associated with the same a into the join $c \leq a$ and let u be the join of all c. Then $\dim u$ is small as indicated and, by monotonicity of lattice terms, we have $p \leq g(\overline{c})$ but $p \not\leq f(\overline{c})$ in $[0, u]$.

In the MOL case, we replace $t(\overline{x})$ by its negation normal form and associate with that a lattice term $h(\overline{y}, \overline{z})$ with unique occurrence of variables such that any y stands for a positive occurence of some x, any z for an occurence of x'. Now, if $0 < t(\overline{a})$, then $p \leq t(\overline{a}) = h(\overline{b}, \overline{c})$ where all b's are a's and all c's are a''s. By the Lemma, $p \leq h(\overline{q}, \overline{r})$ for some substitution by atoms whence $p \leq t(\overline{d})$ for some $\overline{d} \in [0, u]$, u the joins of all q's and r's. Again, $\dim u$ is bounded as stated. □

Theorem 1.3. Huhn, H., Czédli, Hutchinson. *For any division ring D with prime subfield F_p,*

$$\mathsf{Th}_{eq}\{L(V) \mid V \text{ a } D\text{-vector space}\} = \mathsf{Th}_{eq}\{L(F_p^n) \mid n < \infty\}$$

and this equational theory is decidable.

Proof. [7]. $L(F_p^n)$ is a sublattice of $L(D^n)$. Any $L(V_D)$ is a sublattice of $L(V_{F_p})$. The latter is atomistic, whence by Prop.1 in the variety generated by the $L(F_p^n)$. This proves that the varieties coincide. Thus, the set of non-valid identities is recursively enumerable. On the other hand, the quasi-variety generated by the $L(V_D)$ is recursively axiomatizable - using Mal'cev's method of axiomatic correspondence. Thus, the equational theory is recursively enumerable, too. A reasonable decision procedure has been provided by Czédli and Hutchinson [3]. □

For MOLs we define the *von Neumann variety*

$$\mathcal{N} = \mathsf{HSP}\{L(\mathbb{R}^n) \mid n < \infty\}$$

Theorem 1.4. $\mathcal{N} = \mathsf{HSP}\{L(\mathbb{C}^n) \mid n < \infty\}$ *and* $\mathsf{Th}_{eq}\mathcal{N}$ *is decidable.*

Here, we consider the canonical real resp. complex scalar products.

Proof. We have the following embeddings $L(\mathbb{R}^n) \subseteq L(\mathbb{C}^n) \subseteq L(\mathbb{R}^{2n})$. Also, due to Tarksi [22], $\mathsf{Th}L(\mathbb{R}^n)$ is decidable. Now, apply Prop.1. □

2. Interpretation of rings via frames

A (von Neumann) *n-frame* is a system a_{ij} $(1 \leq i, j \leq n)$ of constants and relations such that in a lattice $L(M)$, M a free R-module on generators $e_1, \ldots, e_n$ and R a ring with unit, these relations are satisfied for

$$a_{ii} = e_i R, \quad a_{ij} = a_{ji} = (e_i - e_j)R$$

and, conversely, any system satisfying the relations is, up to isomorphism, this canoncial one. In particular, the ring R can be interpreted into $L(M)$ via $r \mapsto (e_1 - e_2 r)R$.

(1) For any modular L with an n-frame, $n \geq 4$, one obtains a ring on $\{x \in L \mid x \oplus a_2 = a_1 a_2\}$. Von Neumann [19], mimicking the above.
(2) There are terms $t_{ij}(\overline{x})$ such that for an $\overline{a}$ in a modular lattice the $t_{ij}(\overline{a})$ form an n-frame in the interval $[\prod_{ij} t_{ij}(\overline{a}),\ \sum_{ij} t_{ij}(\overline{a})]$. Moreover, $t_{ij}(\overline{a}) = a_{ij}$ if $\overline{a}$ is an n-frame, already. G. Bergman and A. Huhn [13] .
(3) For MOLs this extends to *orthogonal n-frames*: $a_{ii} \leq a'_{jj}$ for $i \neq j$ and $\prod_{ij} a_{ij} = 0$. R. Mayet and M. Roddy [16].

3. Uniform word problem

A *quasi-identity* is a first order sentence of the form

$$\forall \overline{x}.\ (\bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x})) \Rightarrow s(\overline{x}) = t(\overline{x})$$

Solvability of the *uniform word problem* for a class $\mathcal{C}$ of algebraic structures means decidability of the set of quasi-identities valid in $\mathcal{C}$. Let $\mathcal{Q}$ denote the quasi-variety generated by $\mathcal{C}$, and $\mathcal{Q}_L$ the class of lattices embedded into reducts of members of $\mathcal{Q}$.

Theorem 3.1. *If, for some field F, $L(F^n) \in \mathcal{Q}_L$ for all $n < \infty$, then the uniform word problem for $\mathcal{C}$ is unsolvable.*

Proof. Let $\mathcal{S}$ denote the class of all semigroups, $F^{n\times n}$ the ring of all $n \times n$-matrices over F, and F_p the prime subfield.

(1) $L(F^n) \cong \overline{L}(F^{n\times n})$, the lattice of principal right ideals
(2) $\mathsf{Th}_\forall\{F^{n\times n} \mid n < \infty\} = \mathsf{Th}_\forall\{F_p^{n\times n} \mid n < \infty\} = \mathsf{Th}_\forall \mathcal{S}_{fin}$ considering multiplicative semigroups Lipshitz [15]
(3) $\mathsf{Th}_{qid}\mathcal{S} \subseteq \Gamma \subseteq \mathsf{Th}_{qid}\mathcal{S}_{fin}$ for no recursive Γ. Gurevich, Lewis [6].
(4) Interpret $\mathcal{S} \rightarrow$ Rings $\rightarrow$ ML via $F \mapsto F[S]$ and $R \mapsto L$ via frames

$\square$

4. Restricted word problem

The restriced word problem for $\mathcal{C}$ considers in each instance a fixed premise $\bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x})$. For a quasi-variety, this amounts to considering a finite presentation. Unsolvability means the existence of some instance with undecidabe decision problem.

Theorem 4.1. Lipshitz, Hutchinson [15, 14]. *If $L(M) \in \mathcal{Q}_L$ for some free module M on an infinite basis, then the restricted word problem for $\mathcal{C}$ is unsolvable - there is a presentation on 5 lattice generators.*

Indeed, any finitely presented semigroup can be interpreted into some $L(M^n)$.

Theorem 4.2. Cohn, McIntyre. *There is a finitely presented division ring D with unsolvable word problem*

Corollary 4.3. *If $L(D^n) \in \mathcal{Q}_L$ for such D and some $n \geq 4$ then the restricted word problem for $\mathcal{Q}$ is unsolvable.*

Theorem 4.4. Roddy [21]. *The restricted word problem for MOLs is unsolvable - there is a presentation on 3 generators.*

This is based on an intricate construction of a division ring as above admitting a scalar product on some D^n - $n = 14$.

5. Undecidable equational theories

Theorem 5.1. Freese [4]. *The equational theory of all modular lattices is undecidable - 5 and even 4 variables suffice.*

The proof is based on the above division rings, frames, and an ingenious device allowing to force relations via terms in free modular lattices.

Proposition 5.2. *For D as above and $n \geq 3$, $\mathsf{Th}_{eq} L(D^n)$ is undecidable.*

Proof. The terms for an n-frame will either yield an n-frame of $L(D^n)$ or just a single element. In the first case, one has terms giving elements of the ring associated with the frame or else a collapse of the frame. Again, considering relations on those ring elements one has terms enforcing these relations simultaneously - or else a collapse. Also, when applied to elements satisfying the relations, these remain unchanged. $\square$

6. Satisfiability problems

Dealing with modular lattices of finite height we consider 0 and 1 as constants.

Lemma 6.1. *Let L be an ML of height n with an n-frame a_{ij}. For any pair $f(\overline{x}), g(\overline{x})$ of lattice terms one can construct lattice polynomials $f^-(\overline{x})$, $g^+(\overline{x})$ with constants a_{ij} such that the following are equivalent*

(1) $L \models \exists\overline{x}.\ f(\overline{x}) < g(\overline{x})$
(2) $L \models \exists\overline{x}.\ f^-(\overline{x}) = 0\ \&\ g^+(\overline{x}) = 1$

If L admits an involution such that $a_{jj} \leq a'_{ii}$ for $j \neq i$, then one can construct $h(\overline{x}, \overline{y})$ and add

(3) $L \models \exists\overline{x}.\ h(\overline{x}, \overline{x}') = 1$

Construction and identification of the output polynomials, as well as reconstruction of the input terms can all be done in PTIME.

Proof. Define $b_k = \sum_{i\leq k} a_{ii}$, $f_k^- = a_{kk}(b_{k-1} + f)$, and $f^- = \prod_k f_k^-$. Similarly, for g^+. Put $h = g(\overline{x})\tilde{f}(\overline{y})$ where $\tilde{f}$ arises from f^- by interchanging $+$ and $\cdot$ and replacing the constants by the corresponding elements of the dual n-frame canonically associated with the given n-frame. In the case of MOLs this gives rise to a ternary discriminator polynomial on L [9]. □

Theorem 6.2. *Let F be a field and $n \geq 3$.*

(*i*) *With each polyonmial $p(\overline{x})$ over F one can associate latttice terms $p^-(\overline{y})$ and $p^+(\overline{y})$ such that*

$$F \models \exists\overline{x}.\ p(\overline{x}) = 0 \quad \Leftrightarrow \quad L(F^n) \models \exists\overline{y}.\ p^-(\overline{y}) = 0\ \&\ p^+(\overline{y}) = 1$$

(*ii*) *With any pair $s(\overline{y}), t(\overline{y})$ of lattice terms on can associate polynomials $p_1(\overline{x}), \ldots, p_n(\overline{x})$ with integer coefficients such that*

$$L(F^n) \models \exists\overline{y}.\ s(\overline{y}) = 0\ \&\ t(\overline{y}) = 1 \quad \Leftrightarrow \quad F \models \exists\overline{x}.\ p_1(\overline{x}) = \ldots = p_n(\overline{x}) = 0$$

(*iii*) *If F^n admits an inner product Φ then with any ortholattice term $t(\overline{y})$ one can associate integer $p_i(\overline{x})$ such that*

$$L(F^n, \Phi) \models \exists\overline{y}.\ t(\overline{y}) = 1 \quad \Leftrightarrow \quad F \models \exists\overline{x}.\ p_1(\overline{x}) = \ldots = p_n(\overline{x}) = 0$$

All this can be done in PTIME and does not depend on F for polynomials with integer coefficients.

Here, we conceive the $p(\overline{x})$ primarily as terms. But, as far as solvability is concerned, transition to a linear combination of monomials can be done in PTIME - adding variables.

Proof. In (i) use the lattice terms providing an n-frame and the interpretation of F into $L(F^n)$. In (ii) and (iii) replace the (ortho)lattice variables by matrices with variables for elements of F and recall the descriptions of joins, meets, and orthocomplements in $\overline{L}(F^{n\times n})$. Solving $t(\overline{y}) = 1$ amounts to capturing the identity matrix. $\square$

Corollary 6.3. *Let F be a subfield of $\mathbb{R}$ and Φ the canonical scalar product on F^n where $n \geq 3$. Then the following satisfiability problems are polynomially equivalent*

$$\begin{array}{rcll} L(F^n) & \models & \exists \overline{x}.\ f(\overline{x}) < g(\overline{x}) & f \leq g \textit{ lattice terms} \\ L(F^n) & \models & \exists \overline{x}.\ f(\overline{x}) = 0 \ \&\ g(\overline{x}) = 1 & f, g \textit{ lattice terms} \\ L(F^n, \Phi) & \models & \exists \overline{x}.\ t(\overline{x}) = 1 & t \textit{ ortholattice term} \\ F & \models & \exists \overline{x}.\ p(\overline{x}) = 0 & p \textit{ integer polynomial} \end{array}$$

As remarked by George McNulty, decidability of the latter is an open and controversial question for $F = \mathbb{Q}$ [20]. To get p from the p_i put $p = \sum_i p_i^2$.

7. Real complexity

Henceforth, we consider $\mathbb{R}^n$ and $\mathbb{C}^n$ always with the canonical scalar product Φ.

Corollary 7.1. *The decision problems for each single $\mathsf{Th}_{eq}L(\mathbb{R}^n, \Phi)$, $n \geq 3$, als well as for $\mathsf{Th}_{eq}\mathcal{N}$ are polynomially equivalent and $coBP(NP^0_{\mathbb{R}})$-complete. In particular, they are coNP-hard and in PSPACE.*

Here, $BP(NP^0_{\mathbb{R}})$ refers to non-deterministic polynomial time in the Blum-Shub-Smale model of real computation with constants $0, 1$, only, and binary input.

Proof. The equational theory of a class $\mathcal{C}$ is just the complement of the set of sentences $\exists \overline{x}.\ t(\overline{x}) = 1$ satisfiable in some member of $\mathcal{C}$. Thus, the claim about the $L(\mathbb{R}^n, \Phi)$ follows from Cor.6.3 and the fact that feasability of integer polynomials over $\mathbb{R}$ is known to be $BP(NP^0_{\mathbb{R}})$-complete cf. [17]. Also, with Prop.1 it follows that $\mathsf{Th}_{eq}\mathcal{N}$ is in $coBP(NP^0_{\mathbb{R}})$. To prove completeness, we interpret feasability of integer polynomials via 3-frames into $L(\mathbb{R}^{3n}, \Phi)$ for all $n \geq 1$ simultaneously: according to $L(\mathbb{R}^{3n}) \cong L((\mathbb{R}^{n\times n})^3)$ we see the x_i as variables for matrices $A_i \in \mathbb{R}^{n\times n}$. Imposing the relations $A_i = A_i^t$ and $A_iA_j = A_jA_i$, which we can enforce via ortholattice terms to be built into the identity $t(\overline{y}) = 1$, we achieve that the A_i are simultaneously diagonalizable, whence from $p(\overline{A}) = 0$ we obtain a solution in $\mathbb{R}$. The cases $3n+1$ and $3n+2$ are dealt with considering the $3n$-part of the frame. $\square$

8. MOL-representations

Given an inner product space (V,Φ) which is an elementary extension of an unitary space, an *e-unitary representation* of an MOL L is a 0-lattice embedding $\varepsilon : L \to L(V)$ such that

$$\varepsilon(a') = \varepsilon(a)^{\perp} \text{ for all } a \in L.$$

Theorem 8.1. Bruns, Roddy, H. [2, 8, 11]. *For any e-unitary representation of an MOL, there is an atomic MOL $\tilde{L}$ which is a sublattice of $L(V)$ and contains both $\varepsilon(L)$ and $L(V,\Phi)$ as sub-OLs.*

Proposition 8.2. H., Roddy [8, 11]. *$L \in \mathcal{N}$ if L admits an e-unitary representation. For subdirectly irreducibles, the converse holds, too.*

Proof. Prop.1 and the fact, that all sections of fixed finite height have the same first order theory. Conversely, by the Jónsson Lemma we have $L \in \mathsf{HSP}_u\{L(\mathbb{C}^n) \mid n < \infty\}$ and show that representability is preserved. $\square$

9. $*$-regular rings and representations

An associative ring (with or without unit) R is (von Neumann) *regular* if for any $a \in R$ there is a *quasi-inverse* $x \in R$ such that $axa = a$. A $*$-ring is a ring with an involution * as additional operation:

$$(x+y)^* = x^* + y^*, \;\; (xy)^* = y^*x^*, \;\; x^{**} = x.$$

e is a *projection* if $e = e^* = e^2$. A $*$-ring is *$*$-regular* if it is regular and, moreover, *positive*: $xx^* = 0$ only for $x = 0$. Equivalently, for any $a \in R$ there is a (unique) projection e such that $aR = eR$. Examples are the $\mathbb{C}^{n\times n}$ with r^* the adjoint matrix. The projections of a $*$-regular ring with unit form an MOL $\overline{L}(R)$ where $e \leq f \Leftrightarrow e = ef$ and $e' = 1 - e$. Now, $e \mapsto eR$ is an isomorphism of $\overline{L}(R)$ onto the ortholattice of principal right ideals of R and we may use the same notation for both.

Let (V,Φ) be an elementary extension of a unitary space. Denote by ϕ^* the adjoint of ϕ - if it exists. An *e-unitary representation* of a $*$-ring R is a ring embedding $\iota : R \to \mathsf{End}(V)$ such that $\iota(r^*) = \iota(r)^*$ for any $r \in R$.

Proposition 9.1. Giudici [5]. *If $\iota : R \to \mathsf{End}(V)$ is an e-unitary representation of the $*$-regular ring R, then*

$$\varepsilon(eR) = \mathsf{Im}\,\iota(e)$$

is an e-unitary representation of the MOL $\overline{L}(R)$ in (V,Φ).

10. Von Neumann algebras

A *von-Neumann algebra* $\mathbf{M}$ is an unital involutive $\mathbb{C}$-subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators of a separable Hilbert space H with $\mathbf{M} = \mathbf{M}''$ where $\mathbf{A}' = \{\phi \in \mathcal{B}(H) \mid \phi\psi = \psi\phi \quad \forall\phi \in \mathbf{A}\}$ is the *commutant* of $\mathbf{A}$. $\mathbf{M}$ is *finite* if $rr^* = 1$ implies $r^*r = 1$. For such, the projections of $\mathbf{M}$ form a (continuous) MOL $L(\mathbf{M})$. A finite von-Neumann algebra is a *factor* if its center is $\mathbb{C} \cdot 1$. Particular examples of finite factors are the algebras $\mathbb{C}^{n\times n}$ of all complex n-by-n-matrices.

Theorem 10.1. Murray-von-Neumann [18]. *Any finite von-Neumann algebra factor is either isomorphic to* $\mathbb{C}^{n\times n}$ *for some* $n < \infty$ (type I_n) *or contains for any* $n < \infty$ *a subalgebra isomorphic to* $\mathbb{C}^{n\times n}$ (type II_1).

Theorem 10.2. Murray-von-Neumann [18]. *For every finite factor* $\mathbf{M}$*, there is a* $*$*-regular ring* $U(\mathbf{M})$ *of unbounded operators on* H *having* $\mathbf{M}$ *as* $*$*-subring and such that* ϕ^* *is adjoint to* ϕ*. Moreover,* $\mathbf{M}$ *and* $U(\mathbf{M})$ *have the same projections.*

Theorem 10.3. $U(\mathbf{M})$ *admits an e-unitary representation.*

Proof. By the Compactness Theorem, it suffices to consider countable $*$-subrings R of $U(\mathbf{M})$. A representation of R is constructed from the given Hilbert space H. Let H_0 be the intersection of all domains of operators $\phi \in R$. Define, recursively, H_{n+1} as the intersection of H_n and all preimages $\phi^{-1}(H_n)$ where $\phi \in R$. $H_\omega = \bigcap_{n<\omega} H_n$. Due to Murray and von Neumann, all H_n and H_ω are dense in H. It easily follows, that $\varepsilon(\phi) = \phi|H_\omega$ defines a representation. $\square$

Corollary 10.4. $\mathsf{Th}_{eq}\mathcal{N} = \mathsf{Th}_{eq}L(\mathbf{M})$ *for any finite von Neumann algebra factor* $\mathbf{M}$ *of infinite dimension.*

Proof. Observe $L(\mathbf{M}) \cong \overline{L}(U(\mathbf{M}))$ and apply Prop.8.2 and 9.1. $\square$

Corollary 10.5. *For any finite von Neumann algebra factors* $\mathbf{M}$ *and* $\mathbf{N}$

$$U(\mathbf{N}) \in \mathsf{HSP}_u U(\mathbf{M}), \quad U(\mathbf{N}) \in \mathsf{HSP}_u\{\mathbb{C}^{n\times n} \mid n < \infty\}$$

and, analogously, for the projection lattices.

Proof. With suitable choice of quasi-inverse, $*$-regular rings form a congruence distributive variety - the congruence lattice of R is isomorphic to that of $\overline{L}(R)$. The $L(\mathbf{M})$ are simple. Thus, the Jónsson Lemma can be applied. $\square$

A question, raised by Connes and still unanswered, asks whether the Banach-space version of this result is true.

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