On lattices with a compact top congruence

Gillibert

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\[ \text{Con} c L \] denotes the set of all compact (i.e., finitely generated) congruences of \( L \). It is a \((\lor, 0)\)-semilattice.

\( \text{Con} c \) can be extended to a functor.

Let \( f : K \rightarrow L \) be a morphism of lattices. \( \text{Con} c f : \text{Con} c K \rightarrow \text{Con} c L \) is defined by \( \Theta L (\{(f(x), f(y)) \mid (x, y) \in \alpha\}) \).

Given lattices \( K \subseteq L \), we say that \( L \) is a congruence-preserving extension of \( K \) if each congruence of \( K \) extends to a unique congruence of \( L \).

Equivalently \( \text{Con} c f \) is an isomorphism, where \( f : K \hookrightarrow L \) is the inclusion map.
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- $\text{Con}_c$ can be extended to a functor. Let $f: K \to L$ be a morphism of lattices. Put $\Theta_L(\{ (f(x), f(y)) \mid (x, y) \in \alpha \})$.

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Example

- Let $K$ be a lattice of cardinality $\leq \aleph_1$, 

- This cannot be extended to lattices of cardinality $\aleph_2$. 

- Let $V$ be a non-distributive variety of lattices. There is no congruence-permutable lattice $L$ such that $\text{Conc}_K \cong \text{Conc}_L$ (G., Wehrung, 2009). 

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- Let $V$ be a non-distributive variety of lattices. The lattice $F_V(\aleph_1)$ has no congruence-permutable congruence-preserving extension (G., Wehrung, 2009). 

- Let $K$ be a countable locally finite lattice, then $K$ has a relatively complemented congruence-preserving extension (Grätzer, Lakser, and Wehrung, 2000). 

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- Given a variety of lattices $\mathcal{V}$, we denote $\mathcal{V}^b$ the category of lattices in $\mathcal{V}$ with a compact top congruence.
- Morphisms in $\mathcal{V}^b$ are morphisms of lattices $f: K \to L$ such that $(\text{Con}_c f)(1_K) = 1_L$. 
Given a variety of lattices $\mathcal{V}$, denote $\mathcal{V}^{0,1}$ the category of bounded lattices in $\mathcal{V}$. 
Bounded lattices

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![Diagram of bounded lattices]
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Questions

Let $\mathcal{V}$ be a variety of lattices. Are any of the following statements true?

1. For all $K \in \mathcal{V}$, there is $L \in \mathcal{V}_0$ such that $\text{Con}_c K \cong \text{Con}_c L$.
2. All $K \in \mathcal{V}_b$ has a congruence-preserving extension in $\mathcal{V}_0$.

Moreover, if any of those assertions is true, can the construction be made functorial?
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Moreover if any of those assertion is true, can the construction be made functorial?
The finitely generated varieties case

**Theorem**

*Let $\mathcal{V}$ be a finitely generated variety of lattices. The following statements are equivalent.*

1. Each countable lattice $L$ in $\mathcal{V}$ has a congruence-preserving extension in $\mathcal{V}$.
2. Let $K$ be a subdirectly irreducible lattice in $\mathcal{V}$, let $x < y$ in $K$ such that $\Theta_K(x, y) = 1_K$, then $x = 0$ and $y = 1$.
3. $\mathcal{V}_b = \mathcal{V}_0, 1$.

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Example

**Fig.** The lattices $M_3$ and $N_5$. 

```
1 1
x c
0 0
```

Notice that $N_5$ satisfies $(2^2)$, but $M_3$ fails $(2^2)$. 

```
b a
```
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**Fig.**: The lattices $M_3$ and $N_5$.

Notice that $N_5$ satisfies $(2)$, but $M_3$ fails $(2)$. 
Proof of $(2) \iff (3)$

$(2)$ For all subdirectly irreducible lattice $K \in \mathcal{V}$, for all $x < y$ in $K$, if $\Theta_K(x, y) = 1_K$ then $x = 0$ and $y = 1$.

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- Similarly $y$ is the largest element of $L$. 

- Hence $L$ is bounded.
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- Let $\alpha \in M(\text{Con } L)$. 
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Let $\alpha \in M(\text{Con } L)$. So $L/\alpha$ is subdirectly irreducible and $\Theta_{L/\alpha}(x/\alpha, y/\alpha) = 1_{L/\alpha}$.
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- Let $\alpha \in M(\text{Con } L)$. So $L/\alpha$ is subdirectly irreducible and $\Theta_{L/\alpha}(x/\alpha, y/\alpha) = 1_{L/\alpha}$.
- So $x/\alpha$ is smaller than every element of $L/\alpha$. 


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- Let $\alpha \in M(\text{Con } L)$. So $L/\alpha$ is subdirectly irreducible and $\Theta_{L/\alpha}(x/\alpha, y/\alpha) = 1_{L/\alpha}$.
- So $x/\alpha$ is smaller than every element of $L/\alpha$.
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- So $x$ is smaller than every element of $L$. So $L$ has a smallest element $x$. 

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- Similarly $y$ is the largest element of $L$. 

Hence $L$ is bounded.
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Proof by example, $\mathcal{V} = \mathcal{M}_3$. 
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A diagram of $\mathcal{M}_3^b$, with no CP-extension in $\mathcal{M}_3^{0,1}$. 
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and other elements.
Proof by example, $\mathcal{V} = \mathcal{M}_3$.

A diagram of $\mathcal{M}^b_3$, with no CP-extension in $\mathcal{M}^{0,1}_3$.

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We cannot add anything and other elements.
Proof of $(1) \iff (2)$

- There is a diagram in $\mathcal{M}_3^b$ that has no congruence-preserving extension into $\mathcal{M}_3^{0,1}$. 


Proof of $(1) \iff (2)$

- There is a diagram in $\mathcal{M}_3^b$ that has no congruence-preserving extension into $\mathcal{M}_{3,1}^{0,1}$.
- Using a *condensate* construction we obtain a countable lattice $L \in \mathcal{M}_3^b$ that has no congruence-preserving extension into $\mathcal{M}_{3,1}^{0,1}$.
A functor

Theorem

There is a functor \( \Psi : \mathcal{M}_3^b \rightarrow \mathcal{M}_3^{0,1} \),
A functor

Theorem

There is a functor $\Psi: \mathcal{M}_3^b \rightarrow \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. 

Denote $\mathcal{M}_3^{\dagger}$ the full subcategory of finite lattices in $\mathcal{M}_3^b$.

First we define $\Psi$ on lattices $L \in \mathcal{M}_3^{\dagger}$.

We denote $\alpha_L$ the smallest congruence of $L$ such that $L/\alpha_L$ is distributive.

Let $\Psi(L)$ be the product of $L/\alpha_L$ and all quotient of $L$ isomorphic to $M_3$. $\Psi$ can be extended on morphisms in $\mathcal{M}_3^{\dagger}$.

Basically only the morphisms $f: 2 \rightarrow M_3$ can cause problems, change them to the only possible morphism that preserves bounds.
A functor

Theorem

There is a functor $\Psi : M^b_3 \to M^{0,1}_3$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. In particular for all $K \in M^b_3$ there is $L \in M^{0,1}_3$ such that $\text{Con}_c K \cong \text{Con}_c L$. 

Denote $M^b_3\dagger$ the full subcategory of finite lattices in $M^b_3$.

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**Theorem**

There is a functor \( \Psi: \mathcal{M}_3^b \rightarrow \mathcal{M}_3^{0,1} \), such that \( \text{Con}_c \circ \Psi \cong \text{Con}_c \).

In particular for all \( K \in \mathcal{M}_3^b \) there is \( L \in \mathcal{M}_3^{0,1} \) such that \( \text{Con}_c K \cong \text{Con}_c L \).

- Denote \( \mathcal{M}_3^{b\uparrow} \) the full subcategory of finite lattices in \( \mathcal{M}_3^b \).
There is a functor $\Psi : M_3^b \rightarrow M_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. In particular for all $K \in M_3^b$ there is $L \in M_3^{0,1}$ such that $\text{Con}_c K \cong \text{Con}_c L$. 

- Denote $M_3^{b\dagger}$ the full subcategory of finite lattices in $M_3^b$.
- First we define $\Psi$ on lattices $L \in M_3^{b\dagger}$. 
Theorem

There is a functor $\Psi: \mathcal{M}_3^b \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. In particular for all $K \in \mathcal{M}_3^b$ there is $L \in \mathcal{M}_3^{0,1}$ such that $\text{Con}_c K \cong \text{Con}_c L$.

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There is a functor $\Psi : \mathcal{M}_3^b \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$.

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Theorem

There is a functor $\Psi : \mathcal{M}_3^b \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. In particular for all $K \in \mathcal{M}_3^b$ there is $L \in \mathcal{M}_3^{0,1}$ such that $\text{Con}_c K \cong \text{Con}_c L$.

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- $\Psi$ can be extended on morphisms in $\mathcal{M}_3^{b\dagger}$. 
There is a functor $\Psi: \mathcal{M}_3^b \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. In particular for all $K \in \mathcal{M}_3^b$ there is $L \in \mathcal{M}_3^{0,1}$ such that $\text{Con}_c K \cong \text{Con}_c L$.

- Denote $\mathcal{M}_3^{b\dagger}$ the full subcategory of finite lattices in $\mathcal{M}_3^b$.
- First we define $\Psi$ on lattices $L \in \mathcal{M}_3^{b\dagger}$.
- We denote $\alpha_L$ the smallest congruence of $L$ such that $L/\alpha_L$ is distributive.
- Let $\Psi(L)$ be the product of $L/\alpha_L$ and all quotient of $L$ isomorphic to $M_3$.
- $\Psi$ can be extended on morphisms in $\mathcal{M}_3^{b\dagger}$.
- Basically only the morphisms $f: 2 \to M_3$ can cause problems,
A functor

**Theorem**

*There is a functor* \( \Psi : \mathcal{M}_3^b \to \mathcal{M}_3^{0,1} \), *such that* \( \text{Con}_c \circ \Psi \cong \text{Con}_c \).

*In particular for all* \( K \in \mathcal{M}_3^b \) *there is* \( L \in \mathcal{M}_3^{0,1} \) *such that* \( \text{Con}_c K \cong \text{Con}_c L \).

- Denote \( \mathcal{M}_3^{b+} \) *the full subcategory of finite lattices in* \( \mathcal{M}_3^b \).
- First we define \( \Psi \) *on lattices* \( L \in \mathcal{M}_3^{b+} \).
- We denote \( \alpha_L \) *the smallest congruence of* \( L \) *such that* \( L/\alpha_L \) *is distributive.*
- Let \( \Psi(L) \) *be the product of* \( L/\alpha_L \) *and all quotient of* \( L \) *isomorphic to* \( \mathcal{M}_3 \).
- \( \Psi \) *can be extended on morphisms in* \( \mathcal{M}_3^{b+} \).
- Basically only the morphisms \( f : 2 \to \mathcal{M}_3 \) *can cause problems, change them to the only possible morphism that preserves bounds.*
A functor

- We have a functor $\Psi : \mathcal{M}_3^{b\dagger} \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$. 

- Every $\mathcal{L} \in \mathcal{M}_3^{b\dagger}$ is a directed colimit in $\mathcal{M}_3^{b\dagger}$.

- Hence we can extend $\Psi$ to a functor $\mathcal{M}_3^{b\dagger} \to \mathcal{M}_3^{0,1}$ that preserves directed colimits.

- Moreover, as $\text{Con}_c$ preserves directed colimits, $\text{Con}_c \circ \Psi \cong \text{Con}_c$. 
A functor

- We have a functor $\Psi : \mathcal{M}_3^{b\dagger} \to \mathcal{M}_3^{0,1}$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$.
- Every $L \in \mathcal{M}_3^b$ is a directed colimit in $\mathcal{M}_3^{b\dagger}$. 
We have a functor $\Psi : \mathcal{M}^b_3 \to \mathcal{M}^{0,1}_3$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$.

Every $L \in \mathcal{M}^b_3$ is a directed colimit in $\mathcal{M}^b_3$.

Hence we can extend $\Psi$ to a functor $\mathcal{M}^b_3 \to \mathcal{M}^{0,1}_3$ that preserves directed colimits.
We have a functor $\Psi : \mathcal{M}_3^{b\dagger} \to \mathcal{M}_{3,1}^0$, such that $\text{Con}_c \circ \Psi \cong \text{Con}_c$.

Every $L \in \mathcal{M}_3^b$ is a directed colimit in $\mathcal{M}_3^{b\dagger}$.

Hence we can extend $\Psi$ to a functor $\mathcal{M}_3^b \to \mathcal{M}_{3,1}^0$ that preserves directed colimits.

Moreover, as $\text{Con}_c$ preserves directed colimits, $\text{Con}_c \circ \Psi \cong \text{Con}_c$. 

A functor
That is all!

Thank you for your attention

Have you any questions?