A Class of Rings for which the Lattice of Preradicals is not a Set

Rogelio Fernández-Alonso[(1)]
Henry Chimal-Dzul[(1)] Silvia Gavito[(2)]

International Conference on Algebras and Lattices
Prague, Czech Republic

(1) Mathematics Department - Universidad Autónoma Metropolitana
(2) Mathematics Institute - Universidad Nacional Autónoma de México

June, 2010

Index

Notation

We denote by:

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.
- $\mathcal{Z}(R)$, the center of R.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.
- $\mathcal{Z}(R)$, the center of R.
- $\mathcal{I}({}_RR)$ ($\mathcal{I}(R_R)$), the lattice of left (right) ideals of R.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.
- $\mathcal{Z}(R)$, the center of R.
- $\mathcal{I}({}_RR)$ ($\mathcal{I}(R_R)$), the lattice of left (right) ideals of R.
- $R\text{-}\mathrm{Mod}$, the category of left R-modules.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.
- $\mathcal{Z}(R)$, the center of R.
- $\mathcal{I}({}_RR)$ ($\mathcal{I}(R_R)$), the lattice of left (right) ideals of R.
- $R\text{-Mod}$, the category of left R-modules.

Notation

We denote by:

- $\mathcal{OR}$, the class of all ordinals.
- R, an associative ring with 1.
- $\mathcal{Z}(R)$, the center of R.
- $\mathcal{I}({}_RR)$ ($\mathcal{I}(R_R)$), the lattice of left (right) ideals of R.
- $R\text{-}\mathrm{Mod}$, the category of left R-modules.

  For each $M, N \in R\text{-}\mathrm{Mod}$,
- $Hom_R(M, N)$, the abelian group of all homomorphisms $f : M \to N$.
- $End_R(M)$, the ring of all endomorphisms $f : M \to M$.
- $Ext_R(M, N)$, the abelian group of all equivalence classes of extensions of N by M.

Preradicals
Definition

Definition

A *preradical* over the ring R is a functor $\sigma : R\text{-Mod} \to R\text{-Mod}$ such that:

1. $\sigma(M) \leq M$ for all $M \in R\text{-Mod}$.
2. For each $f \in Hom_R(M, N)$, $f(\sigma(M)) \leq \sigma(N)$.

Preradicals

Definition

Definition

A *preradical* over the ring R is a functor $\sigma : R\text{-Mod} \to R\text{-Mod}$ such that:

1. $\sigma(M) \leq M$ for all $M \in R\text{-Mod}$.
2. For each $f \in Hom_R(M, N)$, $f(\sigma(M)) \leq \sigma(N)$.

The class of all preradicals over R is denoted by $R\text{-pr}$.

R-pr as a complete (big) lattice

R-pr as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.

R-pr as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.
- *Join:* $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.

R-pr as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.
- *Join:* $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.
- *Meet:* $(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$.

$R\text{-pr}$ as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.
- *Join:* $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.
- *Meet:* $(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$.

R-pr as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.
- *Join:* $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.
- *Meet:* $(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$.

Remark: R-pr has joins and meets for arbitrary subclasses.

R-pr as a complete (big) lattice

Let $\sigma, \tau \in R\text{-pr}$.

- *Order:* $\sigma \preceq \tau$ if, for every $M \in R\text{-Mod}$, $\sigma(M) \leq \tau(M)$.
- *Join:* $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.
- *Meet:* $(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$.

Remark: R-pr has joins and meets for arbitrary subclasses.

- *Least element:* zero functor, denoted by **0**.
- *Greatest element:* identity functor, denoted by **1**.

Two Operations in R-pr

Idempotent and Radical Preradicals

Two Operations in R-pr

Idempotent and Radical Preradicals

Definition

Let $\sigma, \ \tau \in R\text{-pr}$.

1. *Product:* $(\sigma\tau)(M) = \sigma(\tau(M))$.

Two Operations in R-pr
Idempotent and Radical Preradicals

Definition

Let $\sigma, \ \tau \in R\text{-pr}$.

1. *Product:* $(\sigma\tau)(M) = \sigma(\tau(M))$.
2. *Coproduct:* $(\sigma : \tau)(M)$ is the submodule of M such that $\sigma(M) \leq (\sigma : \tau)(M)$ and $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$.

Two Operations in R-pr

Idempotent and Radical Preradicals

Definition

Let $\sigma, \ \tau \in R\text{-pr}$.

1. *Product:* $(\sigma\tau)(M) = \sigma(\tau(M))$.
2. *Coproduct:* $(\sigma : \tau)(M)$ is the submodule of M such that $\sigma(M) \leq (\sigma : \tau)(M)$ and $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$.

Two Operations in R-pr

Idempotent and Radical Preradicals

Definition

Let $\sigma, \ \tau \in R\text{-pr}$.

1. *Product:* $(\sigma\tau)(M) = \sigma(\tau(M))$.
2. *Coproduct:* $(\sigma : \tau)(M)$ is the submodule of M such that $\sigma(M) \leq (\sigma : \tau)(M)$ and $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$.

Definition

Let $\sigma \in R\text{-pr}$.

1. σ is *idempotent* if $\sigma\sigma = \sigma$.

Two Operations in R-pr
Idempotent and Radical Preradicals

Definition

Let $\sigma, \tau \in R\text{-pr}$.

1. *Product:* $(\sigma\tau)(M) = \sigma(\tau(M))$.
2. *Coproduct:* $(\sigma : \tau)(M)$ is the submodule of M such that $\sigma(M) \leq (\sigma : \tau)(M)$ and $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$.

Definition

Let $\sigma \in R\text{-pr}$.

1. σ is *idempotent* if $\sigma\sigma = \sigma$.
2. σ is a *radical* if $(\sigma : \sigma) = \sigma$.

Powers of a Preradical

Definition

Let $\sigma \in R$-pr. For each $\gamma \in \mathcal{OR}$, σ^{γ} is defined recursively as follows:

- $\sigma^1 := \sigma$.

Powers of a Preradical

Definition

Let $\sigma \in R\text{-pr}$. For each $\gamma \in \mathcal{OR}$, σ^{γ} is defined recursively as follows:

- $\sigma^1 := \sigma$.
- $\sigma^{\gamma+1} := \sigma\sigma^{\gamma}$.

Powers of a Preradical

Definition

Let $\sigma \in R\text{-pr}$. For each $\gamma \in \mathcal{OR}$, σ^{γ} is defined recursively as follows:

- $\sigma^{1} := \sigma$.
- $\sigma^{\gamma+1} := \sigma\sigma^{\gamma}$.
- If γ is a limit ordinal, then $\sigma^{\gamma} := \bigwedge\{\sigma^{\beta} \mid \beta < \gamma\}$.

Powers of a Preradical

Definition

Let $\sigma \in R\text{-pr}$. For each $\gamma \in \mathcal{OR}$, σ^γ is defined recursively as follows:

- $\sigma^1 := \sigma$.
- $\sigma^{\gamma+1} := \sigma\sigma^\gamma$.
- If γ is a limit ordinal, then $\sigma^\gamma := \bigwedge\{\sigma^\beta \mid \beta < \gamma\}$.

Powers of a Preradical

Definition

Let $\sigma \in R$-pr. For each $\gamma \in \mathcal{OR}$, σ^γ is defined recursively as follows:

- $\sigma^1 := \sigma$.
- $\sigma^{\gamma+1} := \sigma\sigma^\gamma$.
- If γ is a limit ordinal, then $\sigma^\gamma := \bigwedge\{\sigma^\beta \mid \beta < \gamma\}$.

Remarks:

- For each $\gamma \in \mathcal{OR}$, $\sigma^\gamma \in R$-pr.
- If $\gamma, \eta \in \mathcal{OR}$ are such that $\gamma < \eta$, then $\sigma^\eta \preceq \sigma^\gamma$.

σ-length of a Module

Definition

Let $\sigma \in R\text{-pr}$ and $M \in R\text{-Mod}$. The *σ-length* of M, denoted by $l_\sigma(M)$, is the least $\lambda \in \mathcal{OR}$ such that

$$\sigma^\lambda(M) = \sigma^{\lambda+1}(M).$$

σ-length of a Module

Definition

Let $\sigma \in R\text{-pr}$ and $M \in R\text{-Mod}$. The *σ-length* of M, denoted by $l_\sigma(M)$, is the least $\lambda \in \mathcal{OR}$ such that

$$\sigma^{\lambda}(M) = \sigma^{\lambda+1}(M).$$

Remark: For each $M \in R\text{-Mod}$, $l_\sigma(M)$ always exists: $\{\sigma^{\lambda}(M)\}_{\lambda \in \mathcal{OR}}$ is a descending chain of submodules of M.

Alpha and Omega Preradicals

Definition

Let $M \in R\text{-Mod}$. A submodule N of M is called *fully invariant* in M (denoted $N \leq_{fi} M$) if for each $f \in End_R(M)$ it follows that $f(N) \leq N$.

Alpha and Omega Preradicals

Definition

Let $M \in R\text{-Mod}$. A submodule N of M is called *fully invariant* in M (denoted $N \leq_{fi} M$) if for each $f \in End_R(M)$ it follows that $f(N) \leq N$.

Definition

Let $M \in R\text{-Mod}$ and $N \leq_{fi} M$. The preradicals α_N^M and ω_N^M are defined as follows. If $K \in R\text{-Mod}$, then:

$$\alpha_N^M(K) = \sum\{f(N) \mid f \in Hom_R(M, K)\},$$

$$\omega_N^M(K) = \bigcap\{f^{-1}(N) \mid f \in Hom_R(K, M)\}.$$

Alpha and Omega Preradicals

Some Properties

Let $\sigma \in R\text{-pr}$.

Alpha and Omega Preradicals

Some Properties

Let $\sigma \in R\text{-pr}$.

Proposition

Let $M, N \in R\text{-Mod}$. Then:

$$\sigma(M) = N \;\Leftrightarrow\; N \leq_{fi} M \text{ and } \alpha^M_N \preceq \sigma \preceq \omega^M_N.$$

Alpha and Omega Preradicals

Some Properties

Let $\sigma \in R\text{-pr}$.

Proposition

Let $M, N \in R\text{-Mod}$. Then:

$$\sigma(M) = N \Leftrightarrow N \leq_{fi} M \text{ and } \alpha_N^M \preceq \sigma \preceq \omega_N^M.$$

Proposition

$$\sigma = \bigvee \{\alpha_{\sigma M}^M \mid M \in R\text{-Mod}\} = \bigwedge \{\omega_{\sigma M}^M \mid M \in R\text{-Mod}\}.$$

Alpha and Omega Preradicals
Some Properties

Let $\sigma \in R\text{-pr}$.

Proposition

Let $M, N \in R\text{-Mod}$. Then:

$$\sigma(M) = N \Leftrightarrow N \leq_{fi} M \text{ and } \alpha_N^M \preceq \sigma \preceq \omega_N^M.$$

Proposition

$$\sigma = \bigvee\{\alpha_{\sigma M}^M \mid M \in R\text{-Mod}\} = \bigwedge\{\omega_{\sigma M}^M \mid M \in R\text{-Mod}\}.$$

Proposition

1. σ is idempotent $\Leftrightarrow \sigma = \bigvee\{\alpha_M^M \mid M \in R\text{-Mod}, \sigma(M) = M\}$.
2. σ is a radical $\Leftrightarrow \sigma = \bigwedge\{\omega_0^M \mid M \in R\text{-Mod}, \sigma(M) = 0\}$.

Radical Modules and Rings
Definition

Definition

Let $M \in R\text{-}\mathrm{Mod}$. We call M a *radical module* if there exist $L \in R\text{-}\mathrm{Mod}$ and a radical $\sigma \in R\text{-}\mathrm{pr}$, $\sigma \neq \mathbf{1}$, such that $M = \sigma(L)$.

Radical Modules and Rings

Definition

Definition

Let $M \in R\text{-}\mathrm{Mod}$. We call M a *radical module* if there exist $L \in R\text{-}\mathrm{Mod}$ and a radical $\sigma \in R\text{-}\mathrm{pr}$, $\sigma \neq \mathbf{1}$, such that $M = \sigma(L)$.

Definition

A ring R is *left radical* if the regular module ${}_R R$ is a radical module.

Radical Rings
A Characterization

Proposition

For a ring R the following conditions are equivalent:

(a) R is left radical.

Radical Rings
A Characterization

Proposition

For a ring R the following conditions are equivalent:

(a) R is left radical.

(b) There is a radical $\sigma \in R\text{-pr}$, $\sigma \neq \mathbf{1}$, such that $\overline{\mathbb{T}}_\sigma = R\text{-Mod}$, where $\overline{\mathbb{T}}_\sigma = \{\sigma(M) \mid M \in R\text{-Mod}\}$.

Radical Rings
A Characterization

Proposition

For a ring R the following conditions are equivalent:

(a) R is left radical.

(b) There is a radical $\sigma \in R\text{-pr}$, $\sigma \neq \mathbf{1}$, such that $\overline{\mathbb{T}}_\sigma = R\text{-Mod}$, where $\overline{\mathbb{T}}_\sigma = \{\sigma(M) \mid M \in R\text{-Mod}\}$.

(c) Every $M \in R$-Mod is a radical module.

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Theorem

If R is a left radical ring, then R-pr is not a set.

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Theorem

If R is a left radical ring, then R-pr is not a set.

Step 1

Let $\sigma \in R\text{-pr}$ be a radical, $\sigma \neq \mathbf{1}$, such that $\overline{\mathbb{T}}_\sigma = R\text{-Mod}$.
There is a sequence $\{M_n\}_{n \geq 1} \subseteq R\text{-Mod}$ such that:

- $M_1 = R$,

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Theorem

If R is a left radical ring, then R-pr is not a set.

Step 1

Let $\sigma \in R\text{-pr}$ be a radical, $\sigma \neq \mathbf{1}$, such that $\overline{\mathbb{T}}_\sigma = R\text{-Mod}$. There is a sequence $\{M_n\}_{n\geq 1} \subseteq R\text{-Mod}$ such that:

- $M_1 = R$,
- $\sigma(M_{n+1}) = M_n$ for each $n \geq 1$.

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Step 2

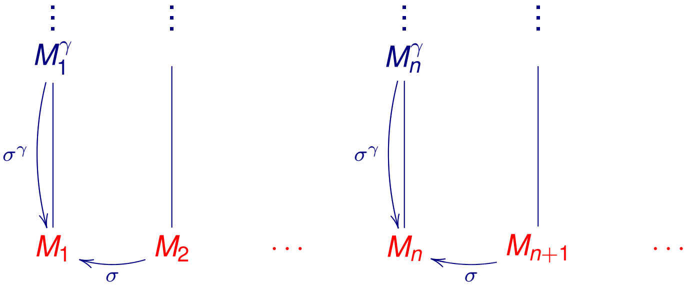
For each $n \geq 1$ and for each $\gamma \in \mathcal{OR}$ there is $M_n^\gamma \in R\text{-Mod}$ such that $\sigma^\gamma(M_n^\gamma) = M_n$.

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Step 2

For each $n \geq 1$ and for each $\gamma \in \mathcal{OR}$ there is $M_n^\gamma \in R$-Mod such that $\sigma^\gamma(M_n^\gamma) = M_n$.



R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Step 3

It follows that, for each $\gamma \in \mathcal{OR}$, $I_\sigma(M_1^\gamma / M_1) = \gamma$.

R-pr is not a Set for Left Radical Rings

Following [Mines, 1971] for $\mathbb{Z}$-Mod

Step 3

It follows that, for each $\gamma \in \mathcal{OR}$, $l_\sigma(M_1^\gamma / M_1) = \gamma$.

Step 4

Thus, there exists a chain of radicals $\{\sigma^\gamma\}_{\gamma \in \mathcal{OR}}$ such that $\sigma^\gamma \neq \sigma^{\gamma+1}$ for all $\gamma \in \mathcal{OR}$. This chain is not a set.

Z-Coinitial Rings
Definition

Definition

Let $\langle P, \leq \rangle$ be a poset. A set $Q \subseteq P$ is *coinitial* in P if for any $x \in P$ there is a $y \in Q$ such that $y \leq x$.

Z-Coinitial Rings
Definition

Definition

Let $\langle P, \leq \rangle$ be a poset. A set $Q \subseteq P$ is *coinitial* in P if for any $x \in P$ there is a $y \in Q$ such that $y \leq x$.

Definition

Let Z be an integral domain. If Z and R are not division rings, we call R a *left (right) Z-coinitial ring* if

Z-Coinitial Rings

Definition

Definition

Let $\langle P, \leq \rangle$ be a poset. A set $Q \subseteq P$ is *coinitial* in P if for any $x \in P$ there is a $y \in Q$ such that $y \leq x$.

Definition

Let Z be an integral domain. If Z and R are not division rings, we call R a *left (right) Z-coinitial ring* if

- Z is a subring of R such that $Z \subseteq \mathcal{Z}(R)$.

Z-Coinitial Rings
Definition

Definition

Let $\langle P, \leq \rangle$ be a poset. A set $Q \subseteq P$ is *coinitial* in P if for any $x \in P$ there is a $y \in Q$ such that $y \leq x$.

Definition

Let Z be an integral domain. If Z and R are not division rings, we call R a *left (right) Z-coinitial ring* if

- Z is a subring of R such that $Z \subseteq \mathcal{Z}(R)$.
- $\{nR \mid n \in Z\}$ is coinitial in the poset $\mathcal{I}({}_R R) \setminus \{0\}$ ($\mathcal{I}(R_R) \setminus \{0\}$);

Z-Coinitial Rings
Definition

Definition

Let $\langle P, \leq \rangle$ be a poset. A set $Q \subseteq P$ is *coinitial* in P if for any $x \in P$ there is a $y \in Q$ such that $y \leq x$.

Definition

Let Z be an integral domain. If Z and R are not division rings, we call R a *left (right) Z-coinitial ring* if

- Z is a subring of R such that $Z \subseteq \mathcal{Z}(R)$.
- $\{nR \mid n \in Z\}$ is coinitial in the poset $\mathcal{I}({}_RR) \setminus \{0\}$ ($\mathcal{I}(R_R) \setminus \{0\}$);

that is, for each $x \in R \setminus \{0\}$ there exist $a \in R$ and $n \in Z \setminus \{0\}$ such that $ax = n$ ($xa = n$).

A Subclass of the Class of Left Radical Rings

Theorem

If R is a **countable**, Z**-coinitial** and **left hereditary** ring, then R is a left radical ring.

A Subclass of the Class of Left Radical Rings

Theorem

If R is a **countable**, Z**-coinitial** and **left hereditary** ring, then R is a left radical ring.

Step 1

$Ext_R(R^{\mathbb{N}}, R)$ is a non zero divisible Z-module.

A Subclass of the Class of Left Radical Rings

Theorem

If R is a **countable**, Z**-coinitial** and **left hereditary** ring, then R is a left radical ring.

Step 1

$Ext_R(R^{\mathbb{N}}, R)$ is a non zero divisible Z-module.

Step 2

$Ext_R(R^{\mathbb{N}}, R)$ has a Z-torsionfree element.

A Subclass of the Class of Left Radical Rings

Following [Mines, 1971] for $R = Z = \mathbb{Z}$

Let $E : 0 \to R \overset{i}{\hookrightarrow} M \to M/R \to 0$ be an exact sequence.

Step 3

The following conditions are equivalent:

(a) $[E] \in Ext_R(M/R, R)$ is Z-torsionfree.

(b) $Hom_R(i, R) : Hom_R(M, R) \to End_R(R)$ is the zero homomorphism.

Step 4

If M/R is cogenerated by R, the following conditions are equivalent:

(b) $Hom_R(i, R) : Hom_R(M, R) \to End_R(R)$ is the zero homomorphism.

(c) $\omega_0^R(M) = R$.

A Subclass of the Class of Left Radical Rings

Following [Mines, 1971] for $R = Z = \mathbb{Z}$

Remarks:

- ω_0^R is a radical.

A Subclass of the Class of Left Radical Rings

Following [Mines, 1971] for $R = Z = \mathbb{Z}$

Remarks:

- ω_0^R is a radical.
- $R^{\mathbb{N}}$ is cogenerated by R.

A Subclass of the Class of Left Radical Rings

Following [Mines, 1971] for $R = Z = \mathbb{Z}$

Remarks:

- ω_0^R is a radical.
- $R^{\mathbb{N}}$ is cogenerated by R.

A Subclass of the Class of Left Radical Rings

Following [Mines, 1971] for $R = Z = \mathbb{Z}$

Remarks:

- ω_0^R is a radical.
- $R^{\mathbb{N}}$ is cogenerated by R.

Step 5

It follows that *R is a left radical ring.*

Some Examples

Corollary

The following rings R are countable, Z-coinitial and left hereditary for some integral domain Z.
Therefore, R-pr is not a set:

Some Examples

Corollary

The following rings R are countable, Z-coinitial and left hereditary for some integral domain Z.
Therefore, $R\text{-pr}$ is not a set:

- $R = \mathbb{Z}$.

Some Examples

Corollary

The following rings R are countable, Z-coinitial and left hereditary for some integral domain Z.
Therefore, $R\text{-pr}$ is not a set:

- $R = \mathbb{Z}$.
- All countable Dedekind domains. For example, $R = F[x]$, with F a countable field.

Some Examples

Corollary

The following rings R are countable, Z-coinitial and left hereditary for some integral domain Z.
Therefore, $R\text{-pr}$ is not a set:

- $R = \mathbb{Z}$.
- All countable Dedekind domains. For example, $R = F[x]$, with F a countable field.
- All countable maximal Z-orders for some Dedekind domain Z. For example, $R = \Lambda \oplus \mathbb{Z}a$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ and $a = \frac{1+i+j+k}{2}$.

Main References

1. T.H. Fay, E.P. Oxford and G.L. Walls, Preradicals in abelian groups, *Houston J. Math* **8(1)** (1982) 39-52.
2. R. Fernández-Alonso, F. Raggi, J. Ríos, H. Rincón and C. Signoret, The lattice structure of preradicals, *Comm. Algebra* **30(3)** (2002) 1533-1544.
3. R. Mines, Radicals and torsion free groups, Seminar Notes, (New Mexico State University, Las Cruces, 1971).
4. R.J. Nunke, Slender groups, *Acta Sci. Math. Szeiged* **23** (1962) 67-73.