Some new classes of ideals in subtraction algebras

A. Yousefian Darani

Department of Mathematics
University of Mohaghegh Ardabili, Iran
yousefian@uma.ac.ir

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Seminar List

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- the composition "o" of functions (and hence (Φ, o) is a function semigroup), and
- the set theoretic subtraction "$\backslash$" (and hence is a subtraction algebra in the sense of Abbott (1969)).

Preliminaries and definitions

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By a subtraction algebra we mean an algebra $(X; -)$ with a single binary operation "-" that satisfies the following identities: for any $x, y, z \in X$,

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(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$.

An order in a subtraction algebra

The subtraction determines an order relation on X:

$$a \leq b \text{ if and only if } a - b = 0$$

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- $a \wedge b = a - (a - b)$;
- The complement of an element $b \in [0, a]$ is $a - b$;
- If $b, c \in [0, a]$, then $b \vee c = (b' \wedge c')' = a - ((a - b) \wedge (a - c)) = a - ((a - b) - (((a - b) - (a - c))))$.

Properties of a subtraction algebra

In a subtraction algebra X, the following are true:

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In a subtraction algebra X, the following are true:

(p1) $(x - y) - y = x - y$.

(p2) $x - 0 = x$ and $0 - x = 0$.

(p3) $(x - y) - x = 0$.

(p4) $x - (x - y) \leq y$.

(p5) $(x - y) - (y - x) = x - y$.

(p6) $x - (x - (x - y)) = x - y$.

(p7) $(x - y) - (z - y) \leq x - z$.

(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(p10) $x, y \leq z$ implies that $x - y = x \wedge (z - y)$.

Definition of an ideal

A nonempty subset A of a subtraction algebra X is called an ideal of X if it satisfies:

(I1) $0 \in A$.

(I2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in A$.

Definition of a prime ideal

Let X be a subtraction algebra. A prime ideal of X is defined to be a proper ideal P of X such that if $x \wedge y \in P$ then $x \in P$ or $y \in P$.

Definition and proposition

Let X be a subtraction algebra, A an ideal of X and S a nonempty subset of X. Set

$$(A :_X S) = \{x \in X | x \wedge s \in A \text{ for every } s \in S\}$$

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- $(A :_X S)$ is an ideal of X and is called the residual of A by S.
- The annihilator of S in X is the set $(0 :_X S)$ and we denote it by $Ann(S)$.

Primal ideals

Definition

Let X be a subtraction algebra and let A be an ideal of X. An element $a \in X$ is called prime to A if

$$a \wedge b \in A \ (b \in X) \Rightarrow b \in A.$$

Denote by $S(A)$ the set of all elements of X that are not prime to A, So

$$S(A) = \{a \in X | a \wedge b \in A \text{ for some } b \in X \backslash A\}$$

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(1) $A \subseteq (A :_X S)$. In particular $A \subseteq (A :_X x)$ for every $x \in X$.
(2) $x \in X$ is prime to A if and only if $A = (A :_X x)$.

Example1

Let $X = \{0, x, y, 1\}$ and define " $-$ " on X by

$-$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

It is easy to check that $(X; -)$ is a subtraction algebra. Then the operation $\wedge$ on X is as follows:

$\wedge$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

Now set $I = \{0, x\}$. Then I is an ideal of X.

$\wedge$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

Now set $I = \{0, x\}$. Then I is an ideal of X.

- The element x is not prime to I since $y \in X \backslash I$ with $x \wedge y = 0 \in I$.

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0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

Now set $I = \{0, x\}$. Then I is an ideal of X.

- The element x is not prime to I since $y \in X \backslash I$ with $x \wedge y = 0 \in I$.
- Also y is prime to I, for if $z \in X$ is such that $y \wedge z \in I$, then $y \wedge z = 0$. Thus either $z = 0$ or $z = x$ both lie in X.

Primeness of adjoint ideal

Lemma

Let X be a subtraction algebra and let A be an ideal of X. If $S(A)$ is a proper ideal of X, then $S(A)$ is a prime ideal of X.

Definition of a primal ideal

Definition

Let X be a subtraction algebra and let A be an ideal of X.

- A is said to be a primal ideal of X provided that $S(A)$ forms an ideal of X. If $S(A)$ is a proper ideal of X, then it is a prime ideal of X, called the adjoint prime ideal P of A. In this case we also say that A is a P-primal ideal of X.

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- X is called a coprimal subtraction algebra provided that the zero ideal of X is primal.

An example of a primal ideal

Example2

Let $X = \{0, 1, 2, 3, 4, 5\}$ and define " $-$ " on X by

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

Then $(X; -)$ is a subtraction algebra. The operation $\wedge$ on X is as follows:

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

Set $A = \{0, 4\}$. Then A is an ideal of X and:

- $S(A) = X$. So A is a primal ideal of X.

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- A is not a prime ideal of X since $3 \wedge 2 = 0 \in A$ but neither 3 nor 2 belong to A. Therefore a primal ideal of X need not be primal. We will prove in a theorem that every prime ideal of X is primal.

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- By (1), for an ideal A of a subtraction algebra X, $S(A)$ need not be a proper ideal of X.

Lemma

Let X be a subtraction algebra and A an ideal of X.
(1) If A is proper, then $A \subseteq S(A)$.
(2) If A is a P-primal ideal of X, then $A \subseteq P$.

Prime⇒ primal

Theorem

Let X be a subtraction algebra. Then every prime ideal of X is primal.

Definition of a zero-divisor

Definition

Let X be a subtraction algebra. An element $a \in X$ is called a zero-divisor of X provided that $a \wedge b = 0$ for some nonzero element $b \in X$.

Is $Z(X)$ an ideal of X?

Let $X = \{0, x, y, 1\}$ and assume that $"-"$ is defined on X as in Example1. Then:

- $Z(X) = \{0, x, y\}$.

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Let X be a subtraction algebra. An element $a \in X$ is called a zero-divisor of X provided that $a \wedge b = 0$ for some nonzero element $b \in X$.

Is $Z(X)$ an ideal of X?

Let $X = \{0, x, y, 1\}$ and assume that "$-$" is defined on X as in Example1. Then:

- $Z(X) = \{0, x, y\}$.
- Since $1 - x = y \in X$, $x \in X$ but $1 \notin X$, so $Z(X)$ is not an ideal of X. This example shows that, for a subtraction algebra X, $Z(X)$ need not necessarily be an ideal of X.

Determining the coprimality via $Z(X)$

Theorem

Let X be a subtraction algebra. Then X is coprimal if and only if $Z(X)$ is an ideal of X.

Weakly prime ideals

Definition

Let X be a subtraction algebra. An ideal P of X is said to be a weakly prime ideal of X if whenever $0 \neq x \wedge y \in P$ then either $x \in P$ or $y \in P$.

Prime $\Rightarrow$ weakly prime but not conversely

Example

Let X be a subtraction algebra.
(1) Every prime ideal of X is weakly prime.
(2) Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then $(X; -)$ is a subtraction algebra. The operation $\wedge$ on X is as follows:

−	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	a	0
b	0	0	b	b	0
c	0	a	b	c	0
d	0	0	0	0	d

Set $P = \{0, b\}$. Then

$-$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	a	0
b	0	0	b	b	0
c	0	a	b	c	0
d	0	0	0	0	d

Set $P = \{0, b\}$. Then

- P is a weakly prime ideal of X since if $0 \neq x \wedge y \in P$, then $x \wedge y = b$. One can check that in any cases either $x = b$ or $y = b$, that is either $x \in P$ or $y \in P$.

$-$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	a	0
b	0	0	b	b	0
c	0	a	b	c	0
d	0	0	0	0	d

Set $P = \{0, b\}$. Then

- P is a weakly prime ideal of X since if $0 \neq x \wedge y \in P$, then $x \wedge y = b$. One can check that in any cases either $x = b$ or $y = b$, that is either $x \in P$ or $y \in P$.
- $c \wedge d = 0 \in P$ while $c \notin P$ and $d \notin P$. Therefore P is not a prime ideal of X.

This example shows that a weakly prime ideal of X need not necessarily be prime.

A characterization for weakly prime ideals

Theorem

Let P be a proper ideal of a subtraction algebra X. Then the following are equivalent:
(i) P is weakly prime.
(ii) For every pair of ideals A and B of X, $0 \neq A \wedge B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Definition

Let X be a subtraction algebra and let A be an ideal of X. An element $a \in X$ is called weakly prime to A if $0 \neq a \wedge b \in A$ ($b \in X$) implies that $b \in A$. We denote by $w(A)$ the set of all elements of X that are not weakly prime to A.

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Let A be a proper ideal of a subtraction algebra X.

- 0 is always weakly prime to A, so $0 \notin w(A)$.
- If $a \in X$ is prime to A, then a is weakly prime to A. Consequently $w(A) \subseteq S(A)$.

Remark

Let A be a proper ideal of a subtraction algebra X.

- 0 is always weakly prime to A, so $0 \notin w(A)$.
- If $a \in X$ is prime to A, then a is weakly prime to A. Consequently $w(A) \subseteq S(A)$.
- $w(0) = \emptyset$ where 0 is the zero ideal of X.

Lemma

Let X be a subtraction algebra and let A be an ideal of X. If $P := w(A) \cup \{0\}$ is an ideal of X, then P is a weakly prime ideal of X.

Definition of weakly primal ideals

Definition

Let X be a subtraction algebra and let A be an ideal of X. A is said to be a weakly primal ideal of X provided that $P := w(A) \cup \{0\}$ forms an ideal of X; this ideal is always a weakly prime ideal, called the weakly adjoint ideal P of A. In this case we also say that A is a P-weakly primal ideal of X.

In this example we show that the concepts "primal ideal" and "weakly primal ideal" are different concepts. Indeed we show that neither imply the other. Let $X = \{0, 1, 2, 3, 4, 5\}$ and define "$-$" on X as in the Example2.

In this example we show that the concepts "primal ideal" and "weakly primal ideal" are different concepts. Indeed we show that neither imply the other. Let $X = \{0, 1, 2, 3, 4, 5\}$ and define "$-$" on X as in the Example2.

Example (Primal $\not\Rightarrow$ weakly primal)

Set $A = \{0, 4\}$. Then, by Example2, A is a primal ideal of X. It is easy to see that $w(A) = \{1, 2, 4\}$. Set $P = w(A) \cup \{0\} = \{0, 1, 2, 4\}$. Since $1 \in P$, $3 - 1 = 0 \in P$ and $3 \notin P$, P is not an ideal of X. So A is not a weakly primal ideal of X. This example shows that a primal ideal need not be weakly primal.

Example (Weakly primal $\nRightarrow$ primal)

Now set $B = \{0, 3\}$. Then B is an ideal of X. Also $S(B) = \{0, 1, 3, 4, 5\}$. Since $1 \in S(B)$, $2 - 1 = 5 \in S(B)$ and $2 \notin S(B)$, $S(B)$ is not an ideal of X. So B is not a primal ideal of X. Moreover $w(B) = \{3\}$. Hence $w(B) \cup \{0\} = B$. So B is a weakly primal ideal of X. This example shows that a weakly primal ideal of X need not be primal.

2-absorbing and weakly 2-absorbing ideals

Definition

A proper ideal A of a subtraction algebra X is said to be a 2-absorbing (resp. weakly 2-absorbing) ideal if whenever $a, b, c \in X$ with $a \wedge b \wedge c \in A$, (resp. $0 \neq a \wedge b \wedge c \in A$) then $a \wedge b \in A$ or $a \wedge c \in A$ or $b \wedge c \in A$.

We can generalize the concept of 2-absorbing ideals in a subtraction algebra X to the concept of (n, m)-absorbing ideals. Suppose that m, n are two positive integers with $n > m$. We say that an ideal A of X is a (n, m)-absorbing ideal if whenever $a_1, a_2, ..., a_n \in X$ and $a_1 \wedge a_2 ... \wedge a_n \in A$, then there are m of a_i's whose meet lies in X. The concept of weakly (m, n)-absorbing ideals is defined in a similar way.

Proposition

Let X be a subtraction algebra and assume that A is an ideal of X. Then

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- Every 2-absorbing ideal of X is weakly 2-absorbing.
- Every prime ideal of X is 2-absorbing.
- Every weakly prime ideal of X is weakly 2-absorbing.
- A is (n, m)-absorbing if and only if it is $(m+1, m)$-absorbing.

Proposition

Let X be a subtraction algebra and assume that A is an ideal of X. Then

- Every 2-absorbing ideal of X is weakly 2-absorbing.
- Every prime ideal of X is 2-absorbing.
- Every weakly prime ideal of X is weakly 2-absorbing.
- A is (n, m)-absorbing if and only if it is $(m+1, m)$-absorbing.
- If A is (n, m)-absorbing, then it is (n, k)-absorbing for every positive integer $k > n$.

Proposition

Let X be a subtraction algebra and assume that A is an ideal of X. Then

- Every 2-absorbing ideal of X is weakly 2-absorbing.
- Every prime ideal of X is 2-absorbing.
- Every weakly prime ideal of X is weakly 2-absorbing.
- A is (n, m)-absorbing if and only if it is $(m+1, m)$-absorbing.
- If A is (n, m)-absorbing, then it is (n, k)-absorbing for every positive integer $k > n$.
- A is a prime ideal if and only if it is a $(2, 1)$-absorbing ideal.

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Let X be a subtraction algebra and assume that A is an ideal of X. Then

- Every 2-absorbing ideal of X is weakly 2-absorbing.
- Every prime ideal of X is 2-absorbing.
- Every weakly prime ideal of X is weakly 2-absorbing.
- A is (n, m)-absorbing if and only if it is $(m+1, m)$-absorbing.
- If A is (n, m)-absorbing, then it is (n, k)-absorbing for every positive integer $k > n$.
- A is a prime ideal if and only if it is a $(2, 1)$-absorbing ideal.
- A is a 2-absorbing ideal if and only if it is a $(3, 2)$-absorbing ideal.

Examples of 2-absorbing and weakly 2-absorbing

Theorem

Let X be a subtraction algebra.

- If P_1 and P_2 are distinct prime ideals of X, then $P_1 \cap P_2$ is a 2-absorbing ideal of X.

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Theorem

Let X be a subtraction algebra.

- If P_1 and P_2 are distinct prime ideals of X, then $P_1 \cap P_2$ is a 2-absorbing ideal of X.
- If P_1 and P_2 are distinct weakly prime ideals of X, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of X.

Thank you for your attention.