The Jordan-Hölder theorem with uniqueness for groups and semimodular lattices*

Gábor Czédli and E. Tamás Schmidt

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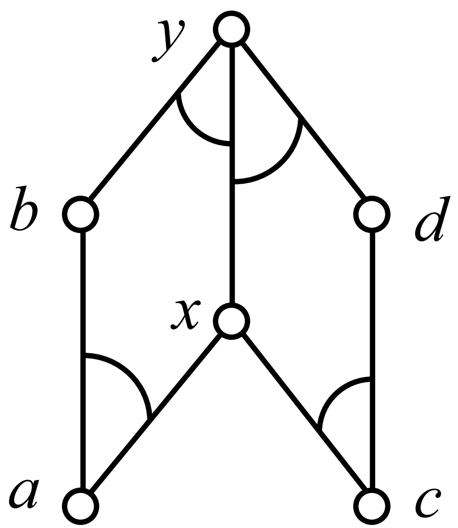
Hint: the subnormal subgroups of a finite (or finite composition length) group form a dually semimodular lattice by Wielandt 1939.

From now on, we are in a semimodular lattice L.

$[a,b] \nearrow\!\searrow [c,d]$ iff $[a,b] \nearrow [x,y]$ and $[x,y] \searrow [c,d]$ for some interval $[x,y]$, that is,



1. Theorem (Main). *Assume that L is semimodular, and*

$$C = \{0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1\} \text{ and}$$

$$D = \{0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1\}. \text{ Then}$$

- $n = m$, *and there is a permutation* π *of the set* $\{1, \ldots, n\}$ *such that the interval* $[c_{i-1}, c_i]$ *is* **up-and-down** *projective to the interval* $[d_{\pi(i)-1}, d_{\pi(i)}]$, *for all* i. *(Jordan-Hölder theorem for sm lattices* + **G**

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- *moreover, this permutation π is* uniquely determined*, and*

- *if $i, j \in \{1, \ldots, n\}$ and $[c_{i-1}, c_i] \nearrow\searrow [d_{j-1}, d_j]$, then $j \leq \pi(i)$.*

(The 3rd part implies the second one (easy exercise).)

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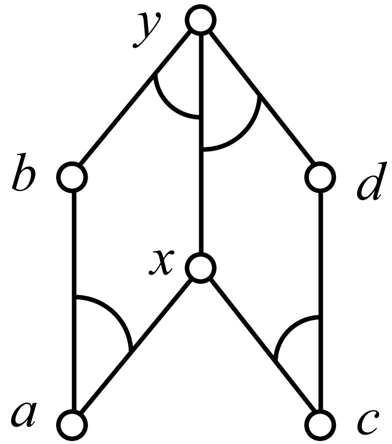
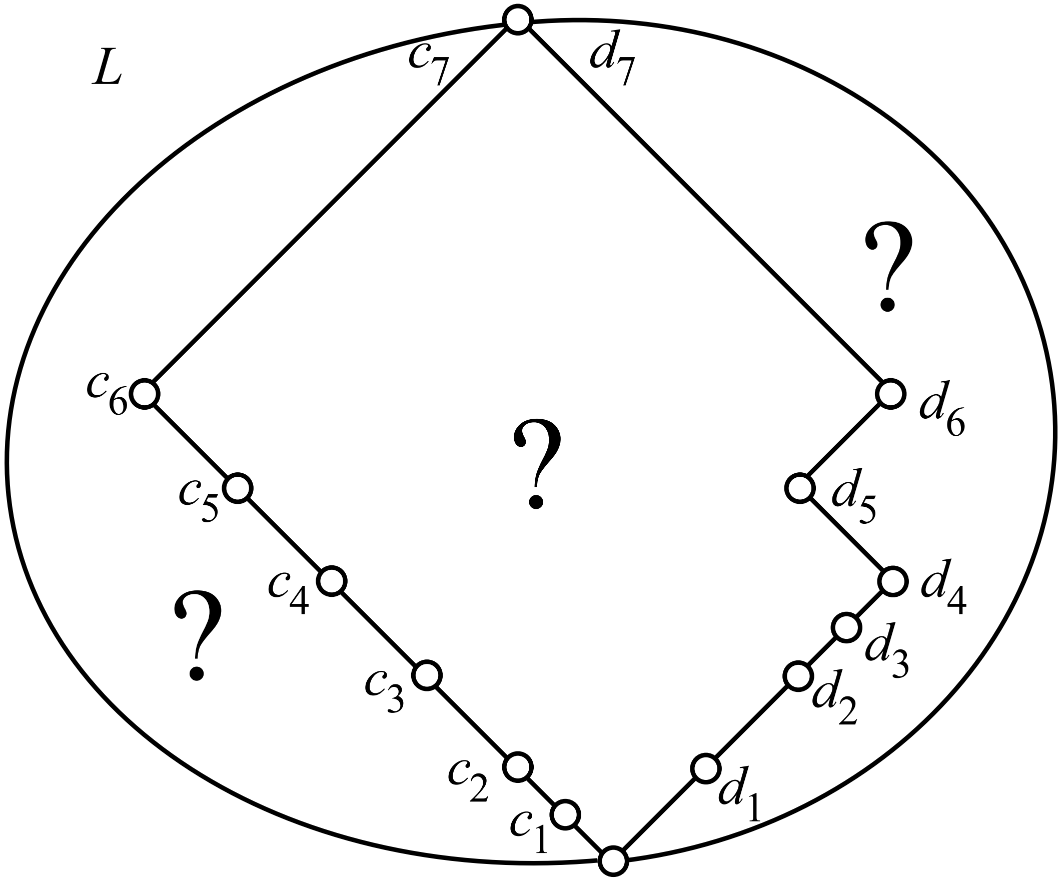
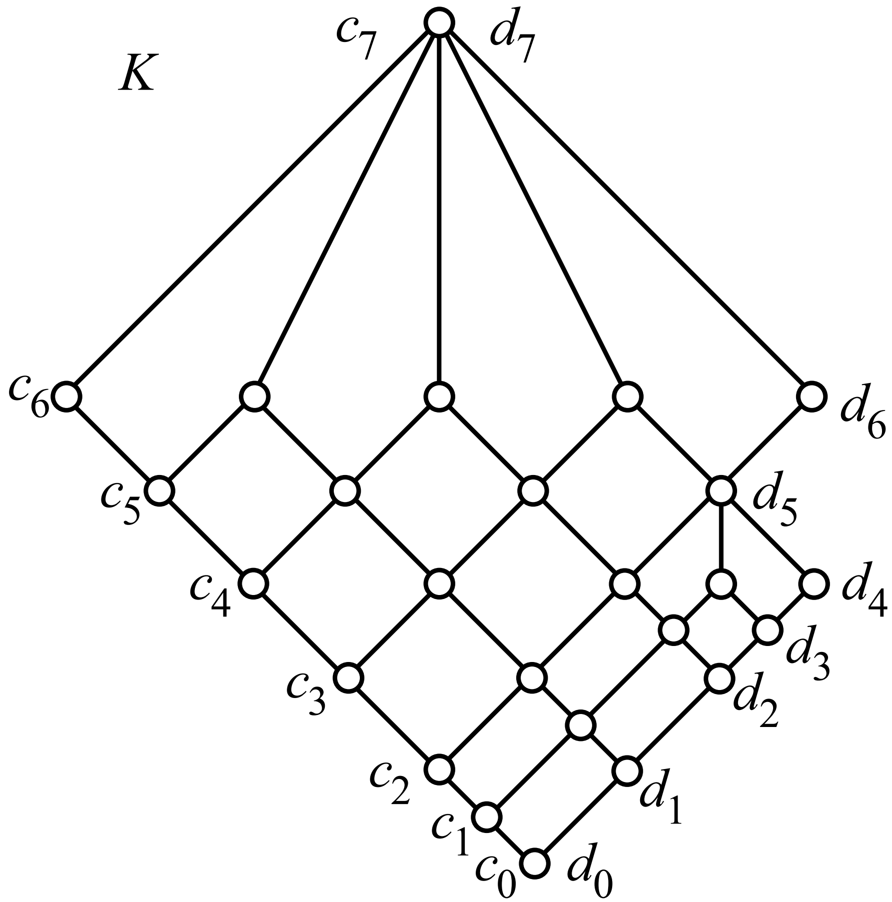
• Let $[a, b]$, $[x, y]$, and $[c, d]$ be **prime** intervals. Then the validity of $[a, b] \nearrow [x, y]$ and $[x, y] \searrow [c, d]$ depends only on $\vee$! E.g.,



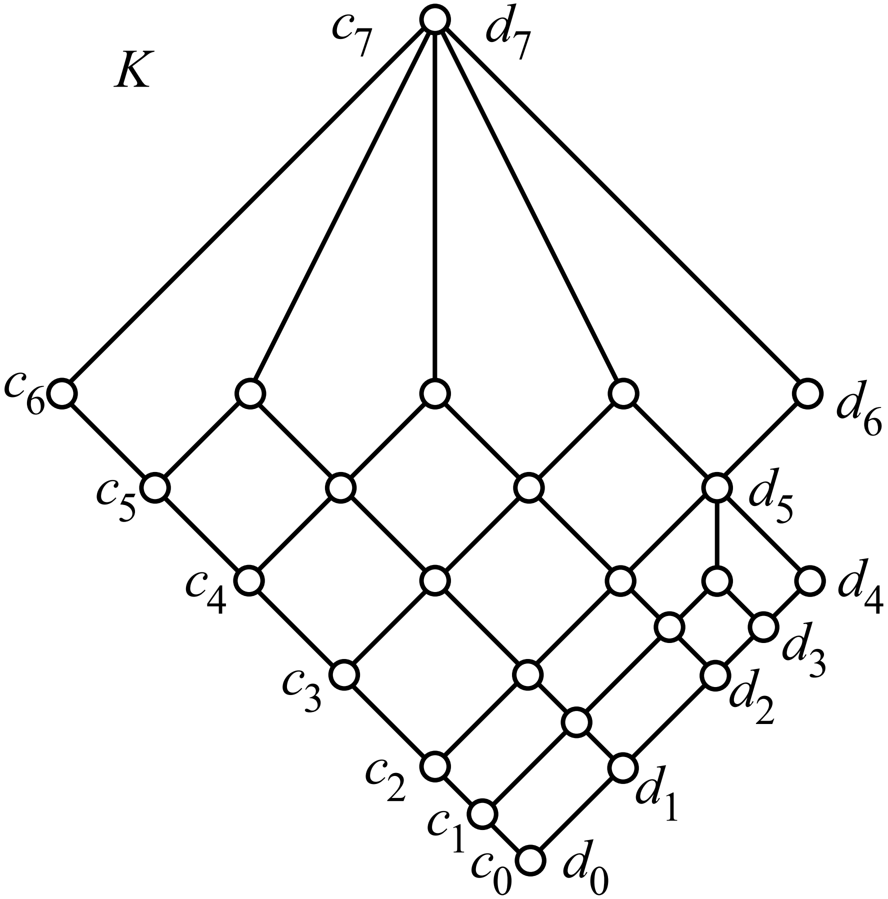
$[a, b] \nearrow [x, y]$ iff $b \vee x = y$ (trivial exercise).

- So, take the **join**-subsemilattice K generated by $C \cup D$!





We get a **slim** (terminology from G. Grätzer and E. Knapp)



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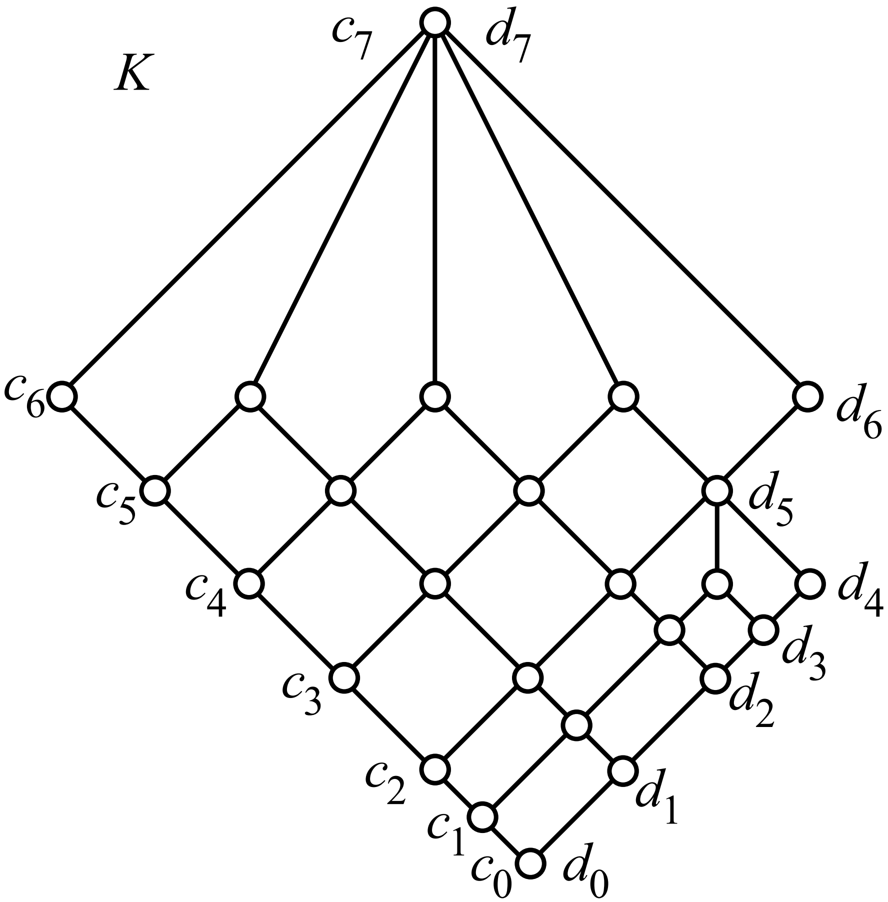

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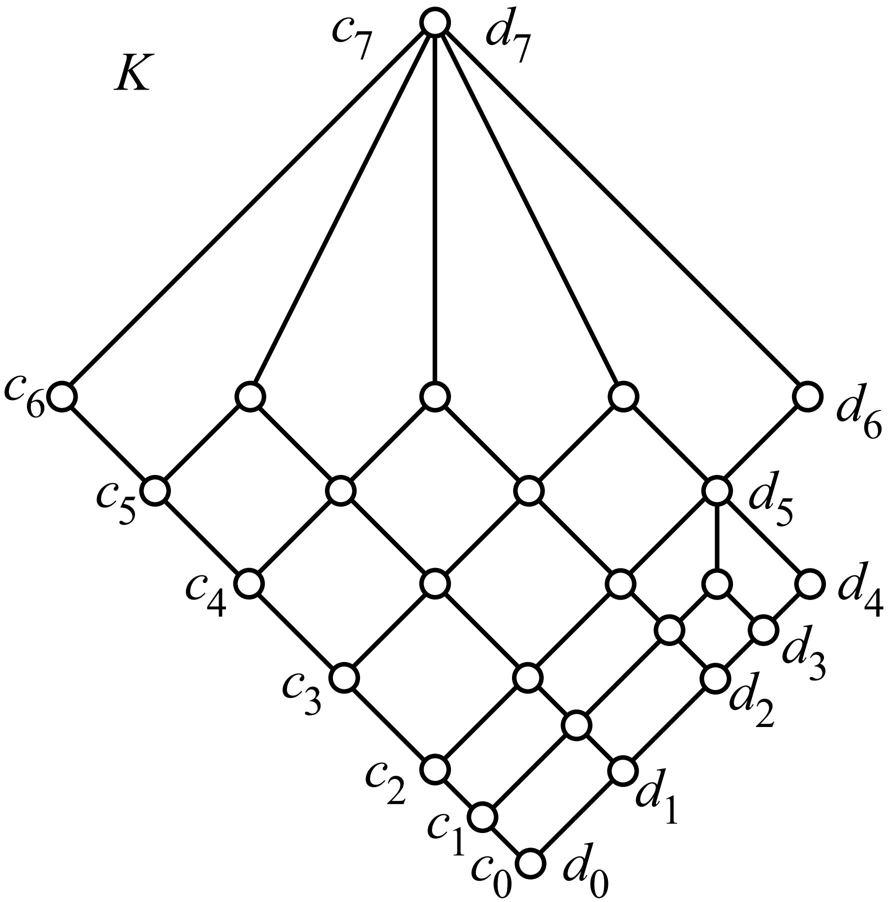

We get a **slim** (terminology from G. Grätzer and E. Knapp) **planar** semimodular lattice K, with left boundary chain C and right boundary chain D. (Straightforward.)

- (Up-and-down) projectivity between **prime**

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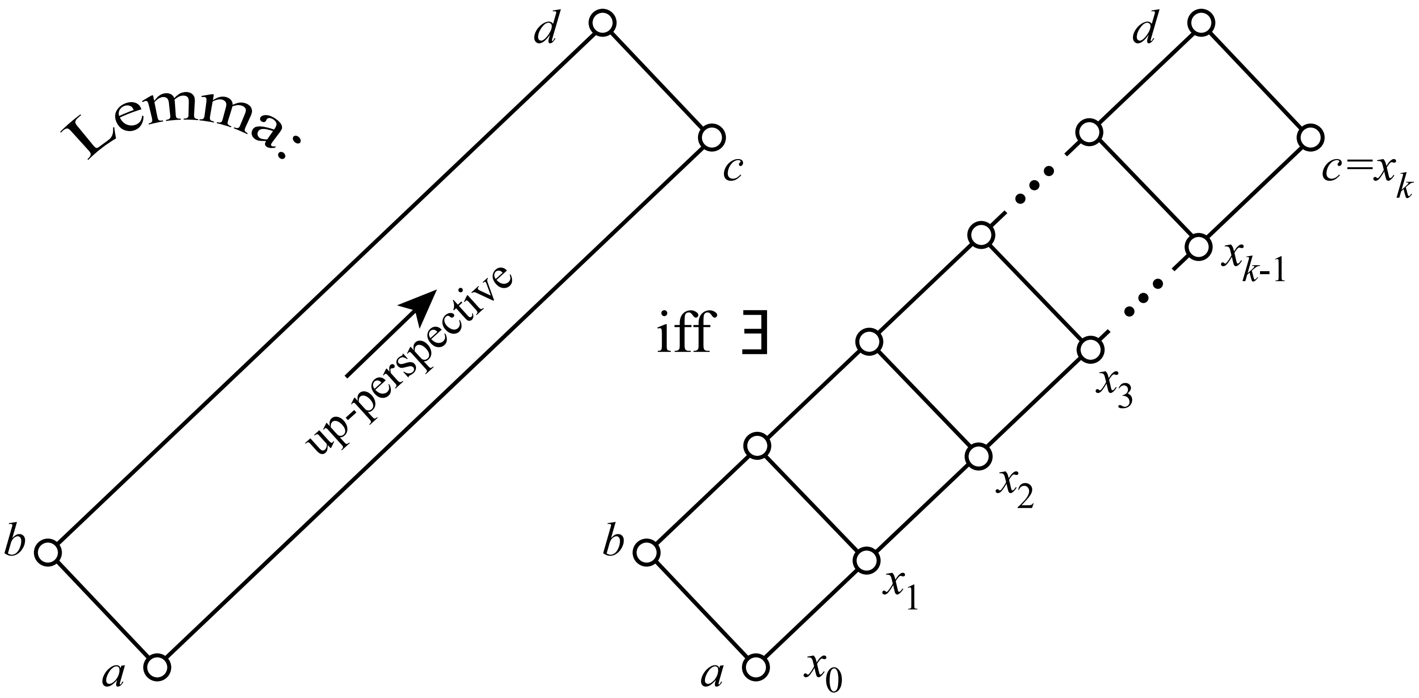

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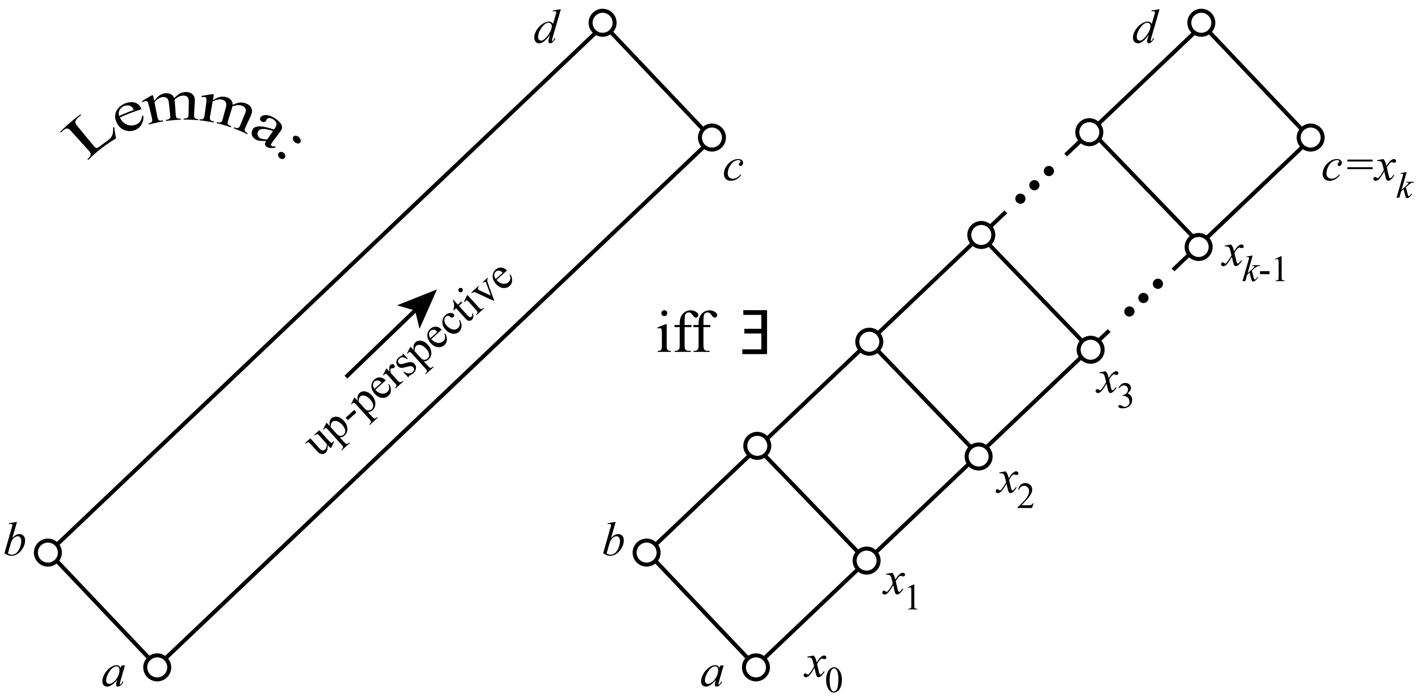
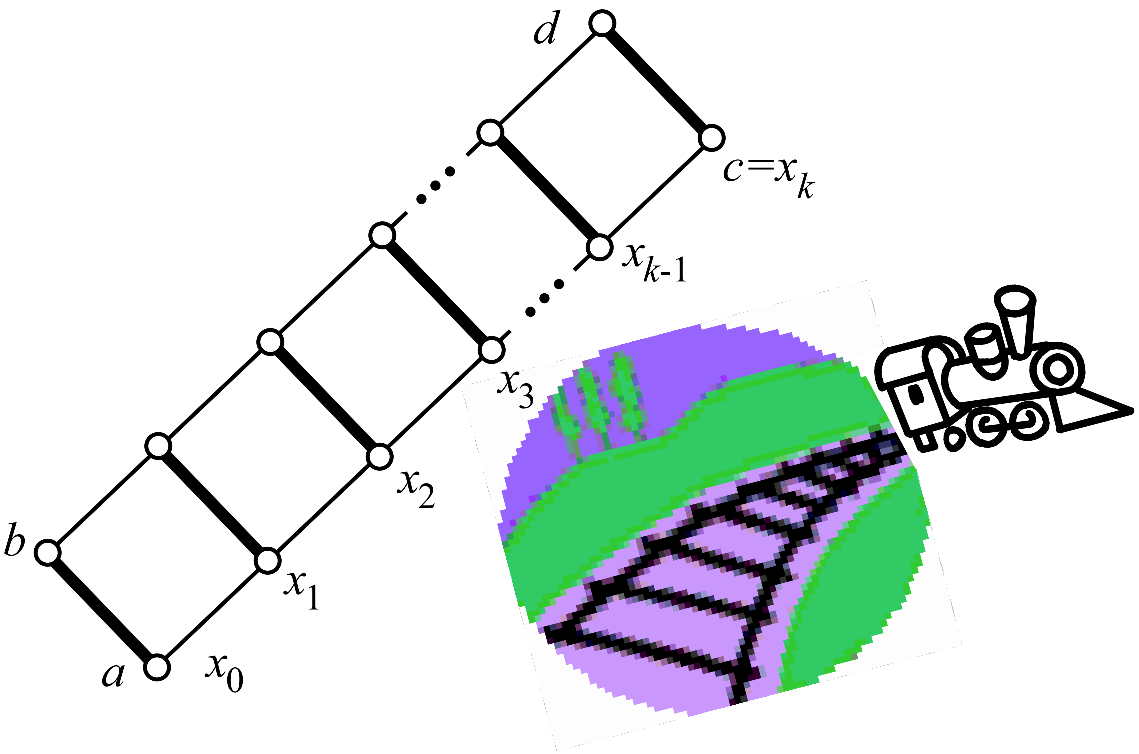


(Straightforward; any maximal chain in $[a, c]$ will do.)

- Let us call it **Locomotive Lemma**, because:



The consecutive prime intervals form a **trajectory**. More precisely: trajectory = class of the equivalence „prime" projectivity described by the Lemma.

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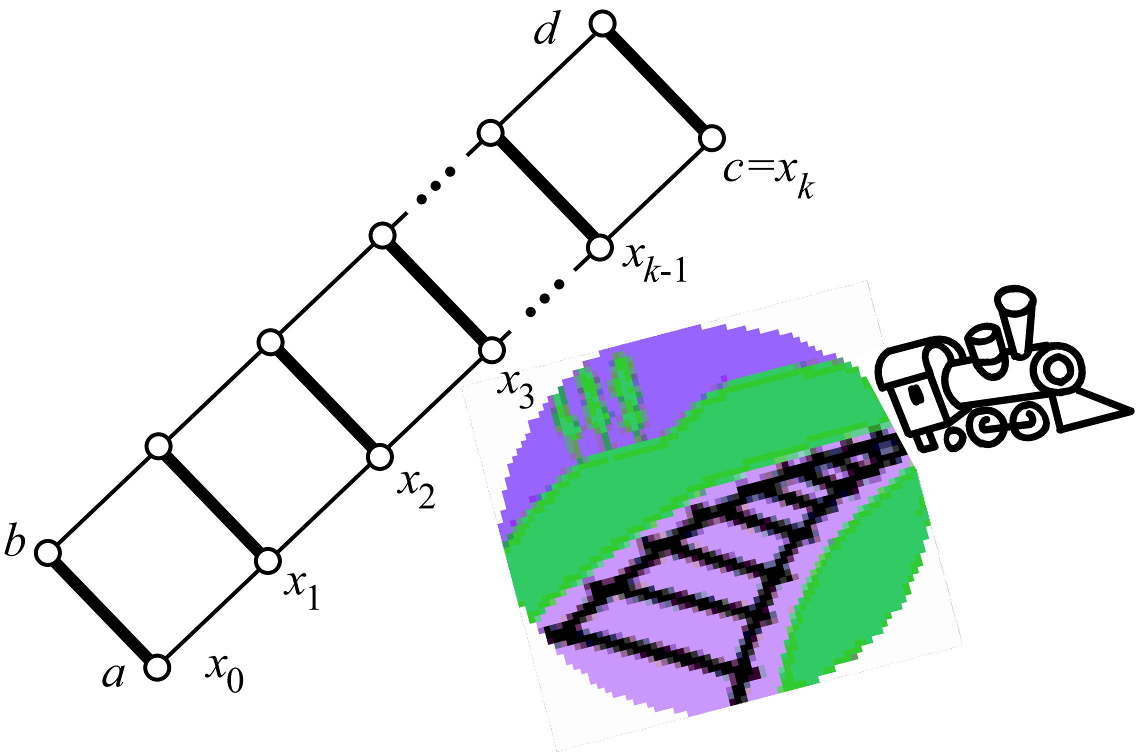
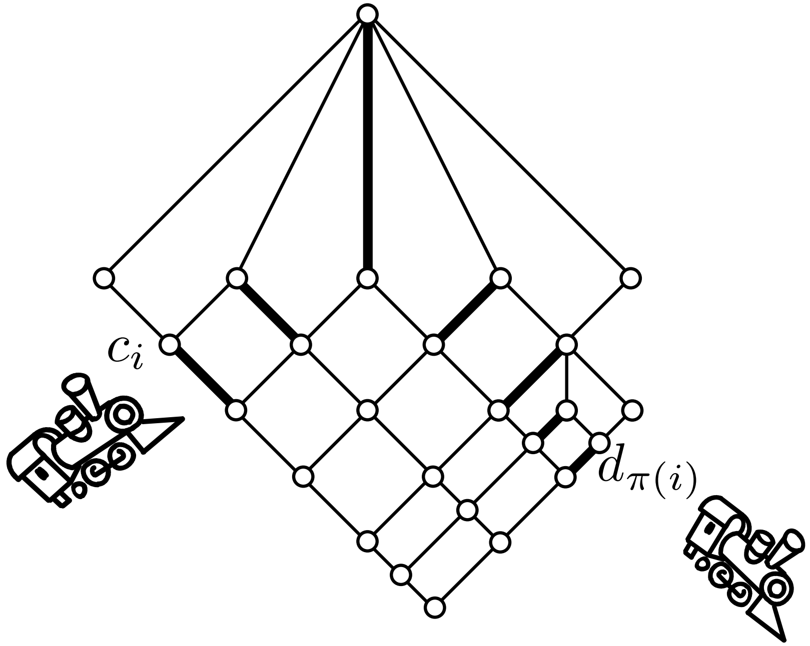
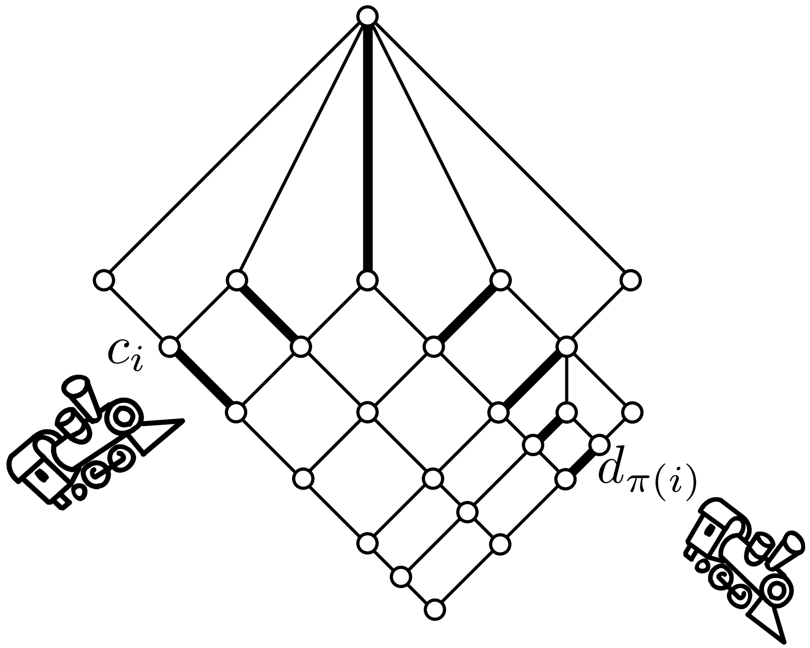


The consecutive prime intervals form a **trajectory**. More precisely: trajectory = class of the equivalence „prime” projectivity described by the Lemma.

Trajectory = railroad of a locomotive. Locomotives will always go from left to right.

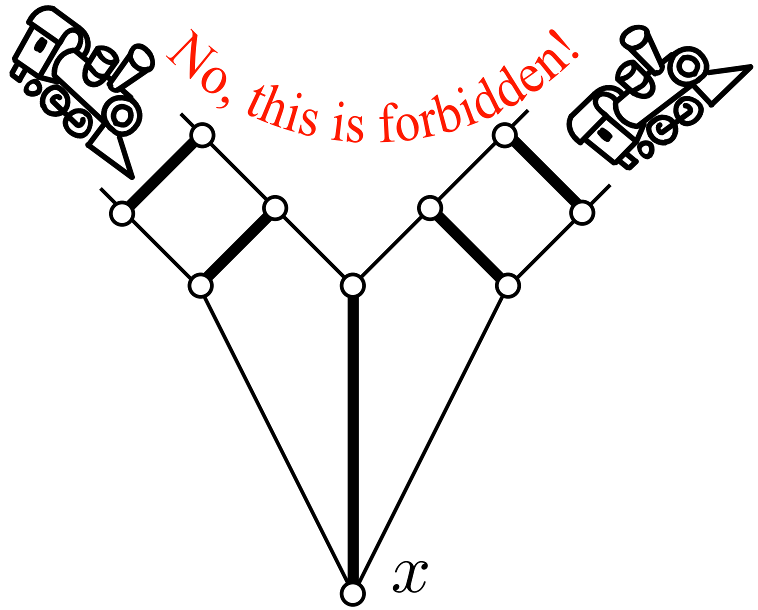


• Since opposite sides of covering squares (= cells) are uniquely determined, e



• Since opposite sides of covering squares (= cells) are uniquely determined, each prime interval of K belongs to a **unique** trajectory. In a trajectory, there is no fork from left to right, neither from right to left; trajectories **never ramify**.

We will think that trajectories (locomotives) go from left to right.



In each moment, the locomotive (on a trajectory) goes either to the northeast, or to the southeast.

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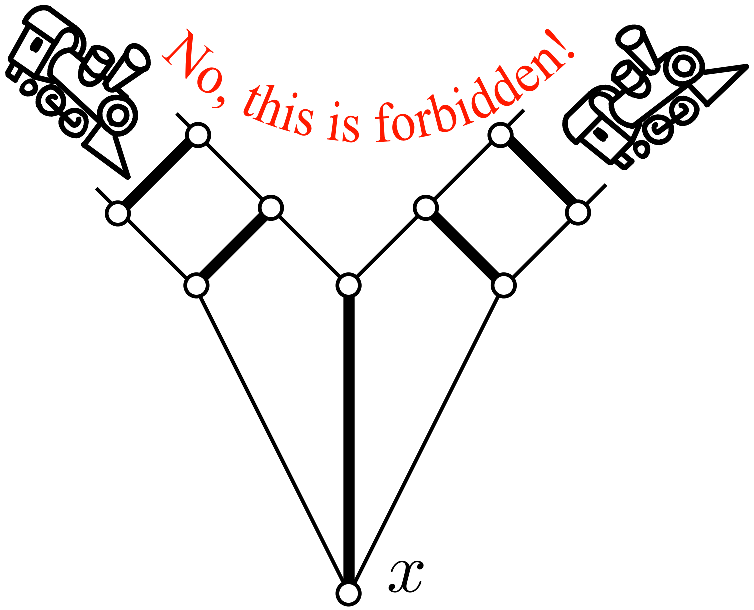
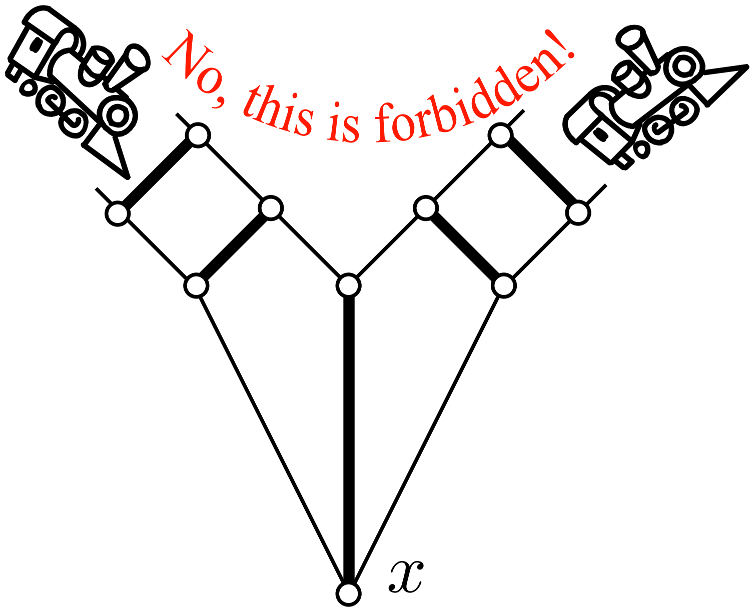


In each moment, the locomotive (on a trajectory) goes either to the northeast, or to the southeast.

- **But once it goes to the southeast, it cannot turn to the northeast later.**



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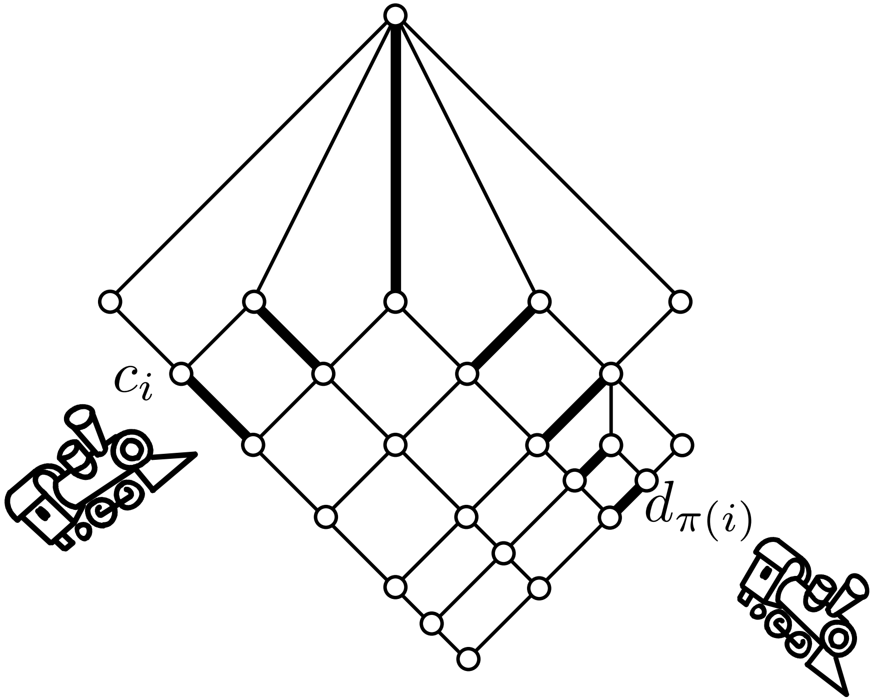
Indeed, otherwise x would have three upper covers, and slimness would easily lead to a contradiction (easy exercise).



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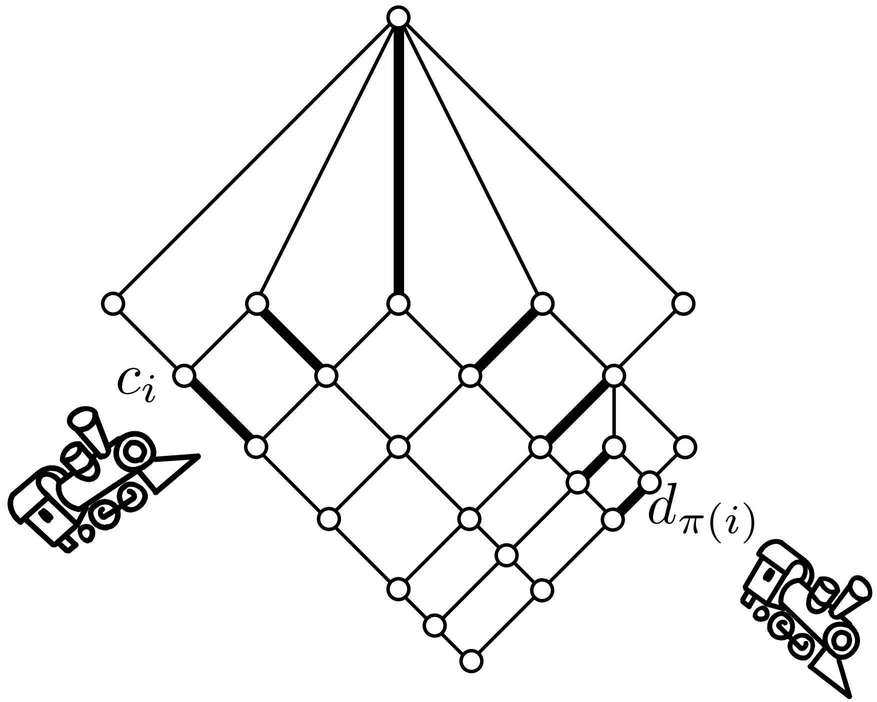
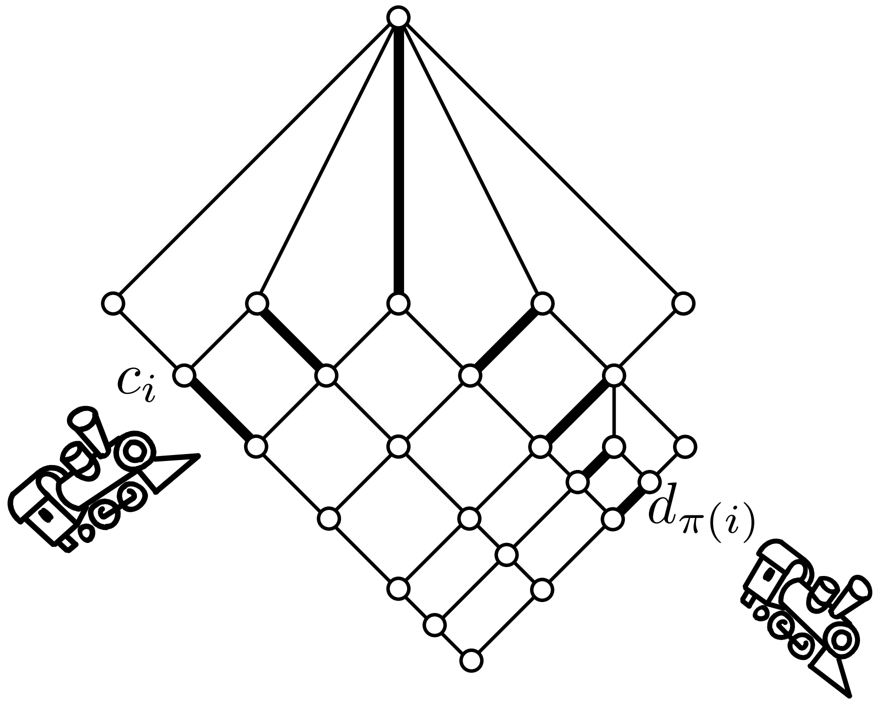


• The previous properties of the trajectories guarantee that matching the (top element of the) „departure” prime interval with the (top element of the) „arrival” interval of the locomotive



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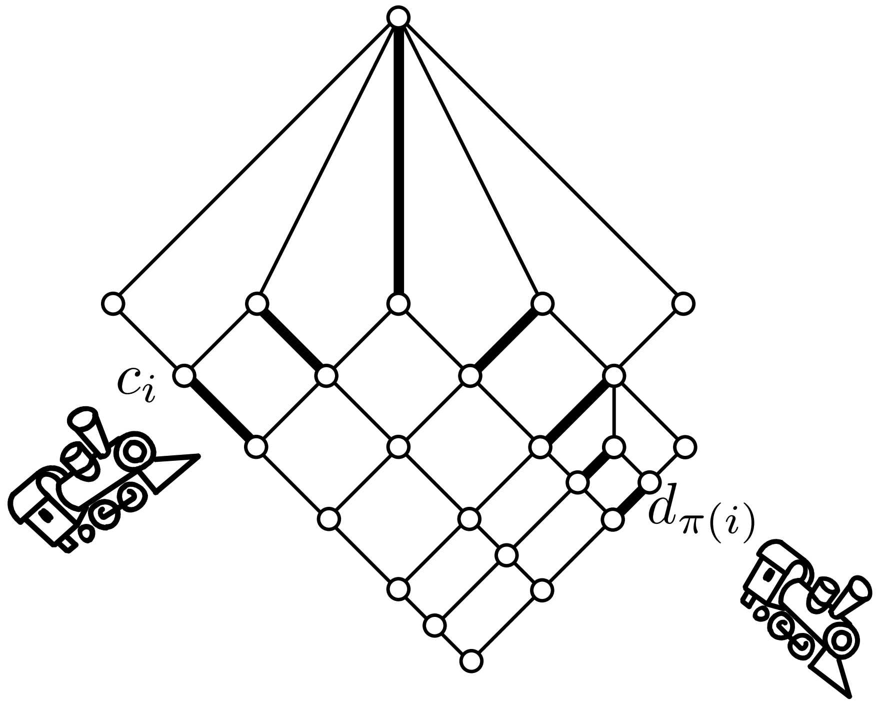

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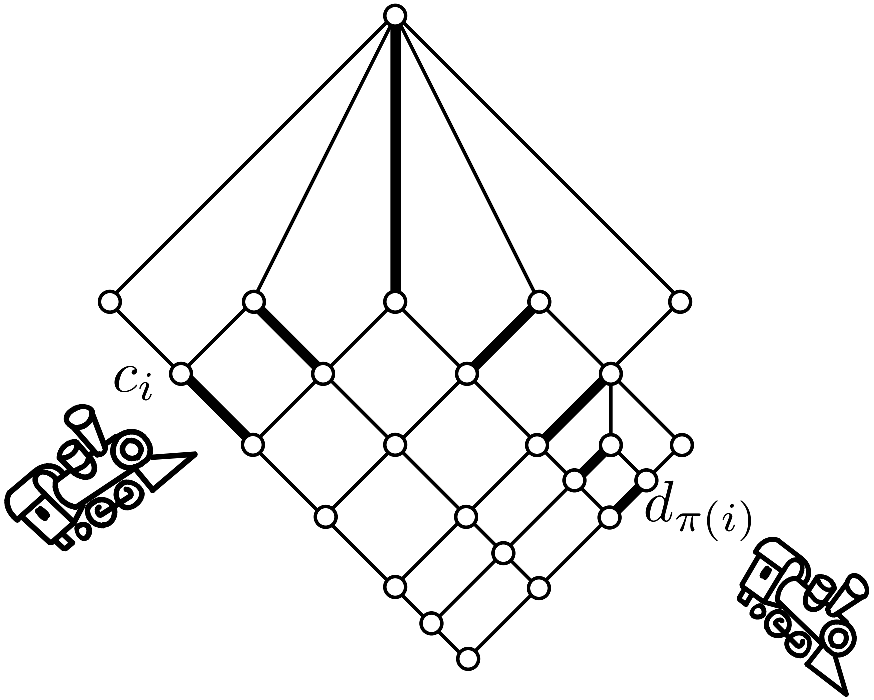

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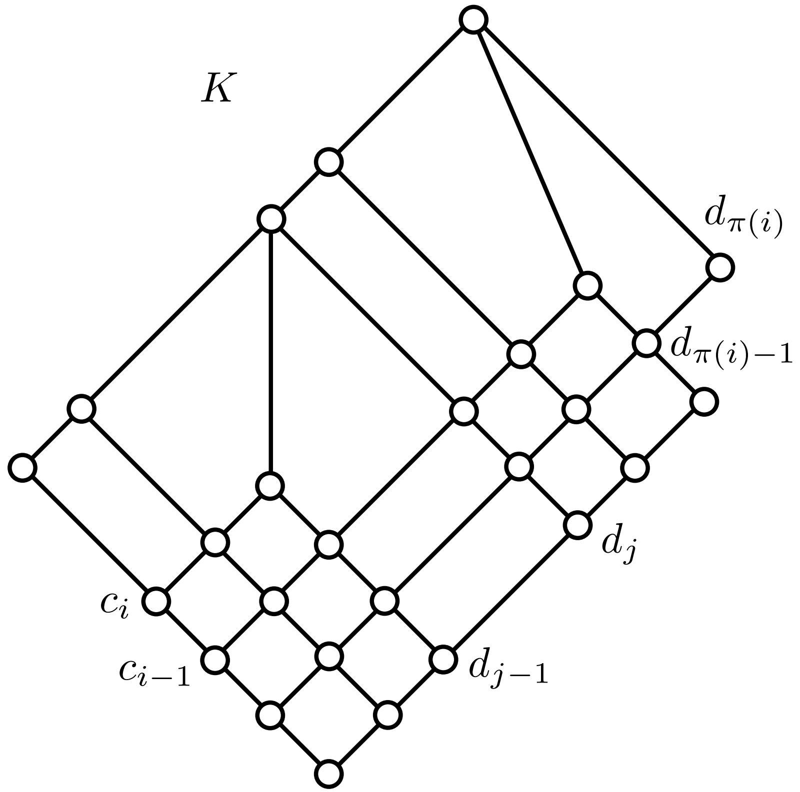
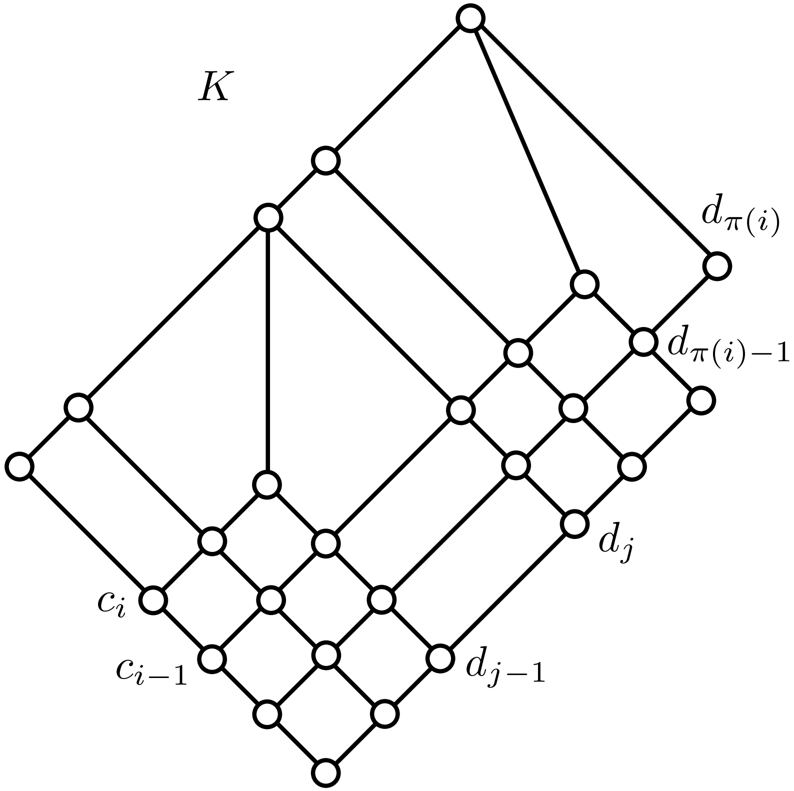


To show $[c_{i-1}, c_i] \nearrow\searrow [d_{j-1}, d_j] \Rightarrow j \leq \pi(i)$, assume $j \neq \pi(i)$.



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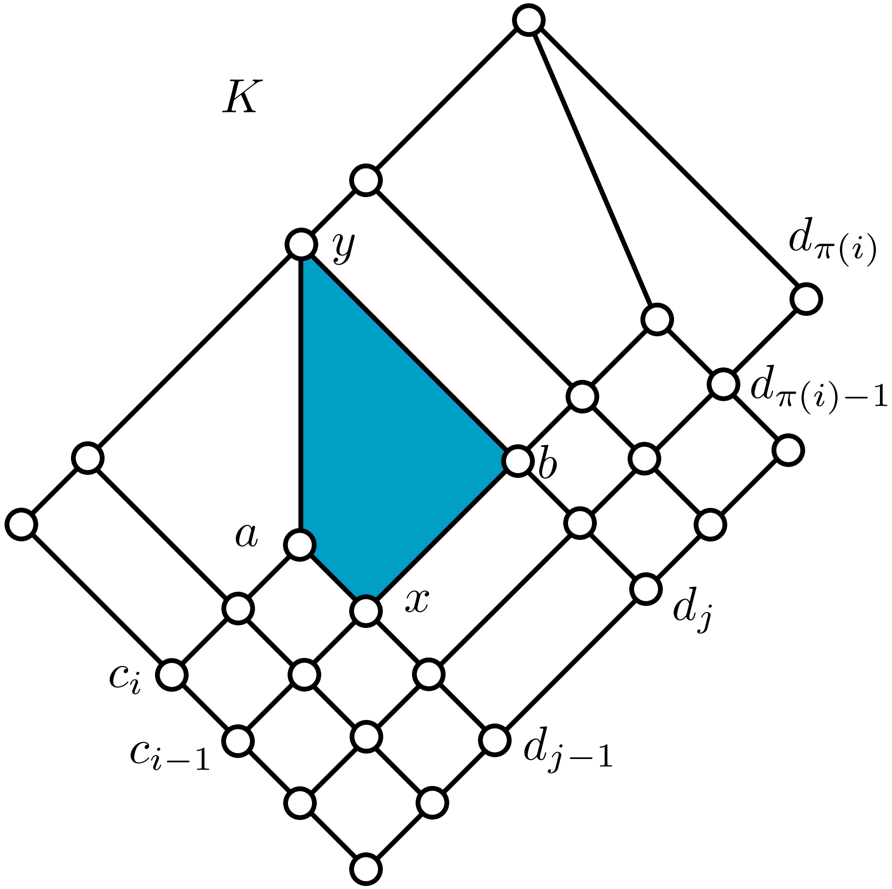
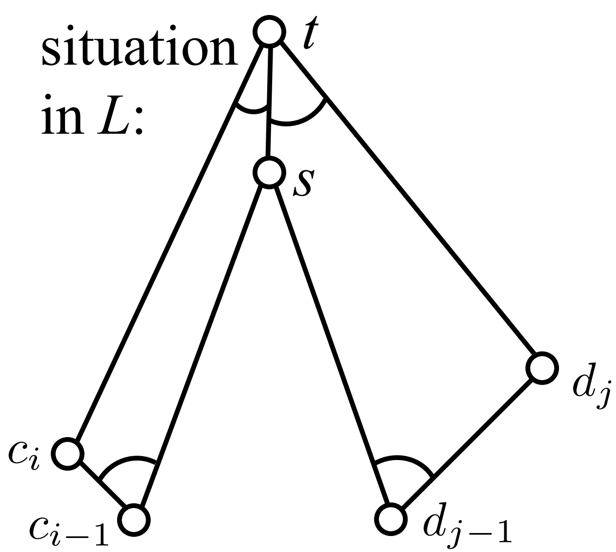
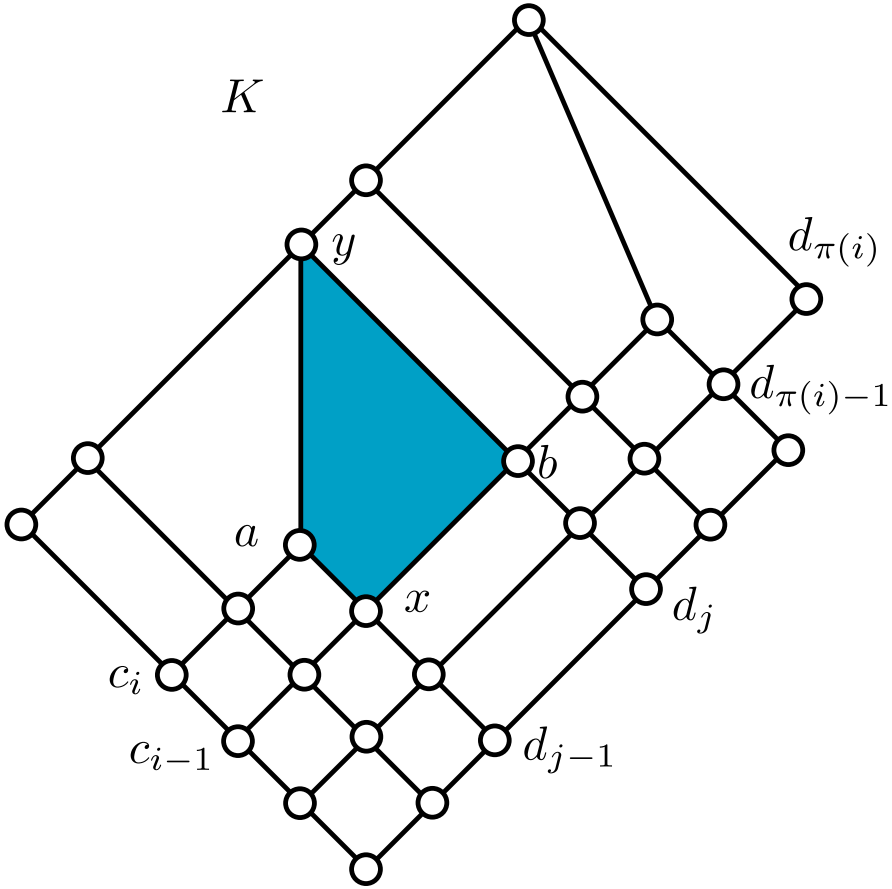
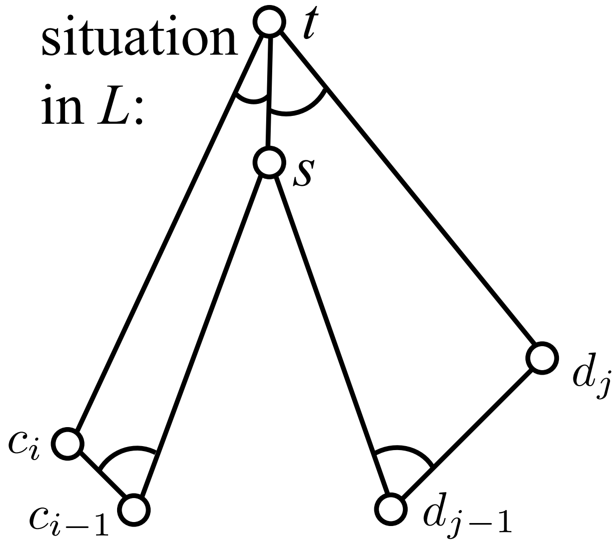
Since K is governed by trajectories that say „$\pi(i)$" and $j \neq \pi(i)$, we know that $[c_{i-1}, c_i] \nearrow\!\searrow [d_{j-1}, d_j]$ holds only in L but not in K.





Denote the joins of c_{i-1} and c_i by d_{j-1}, d_j by x, y, a, b.





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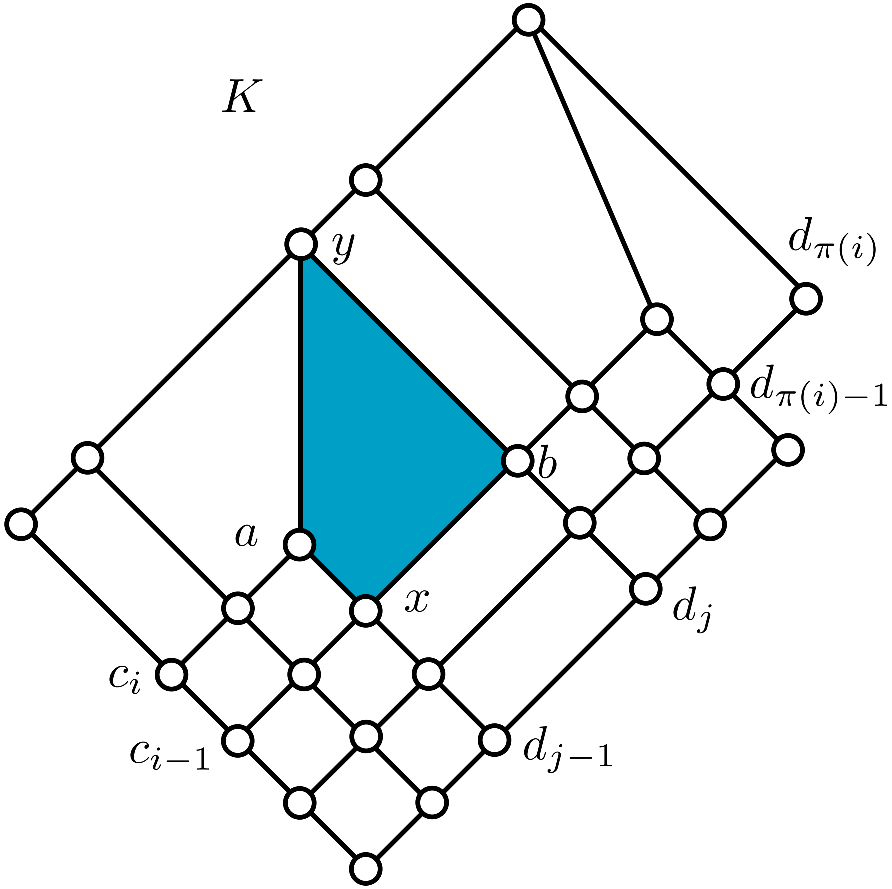
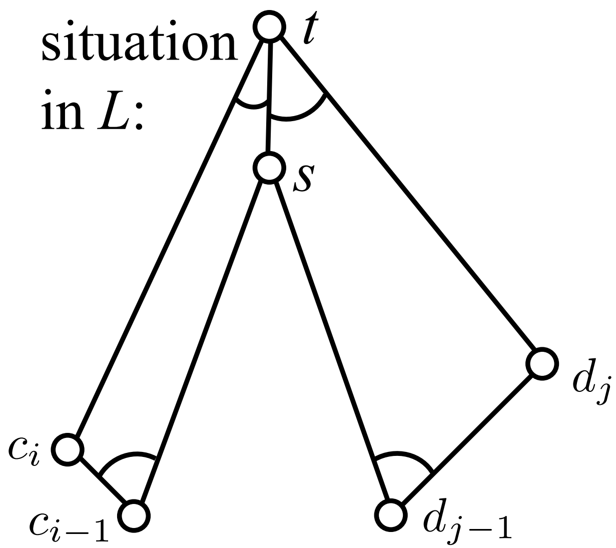
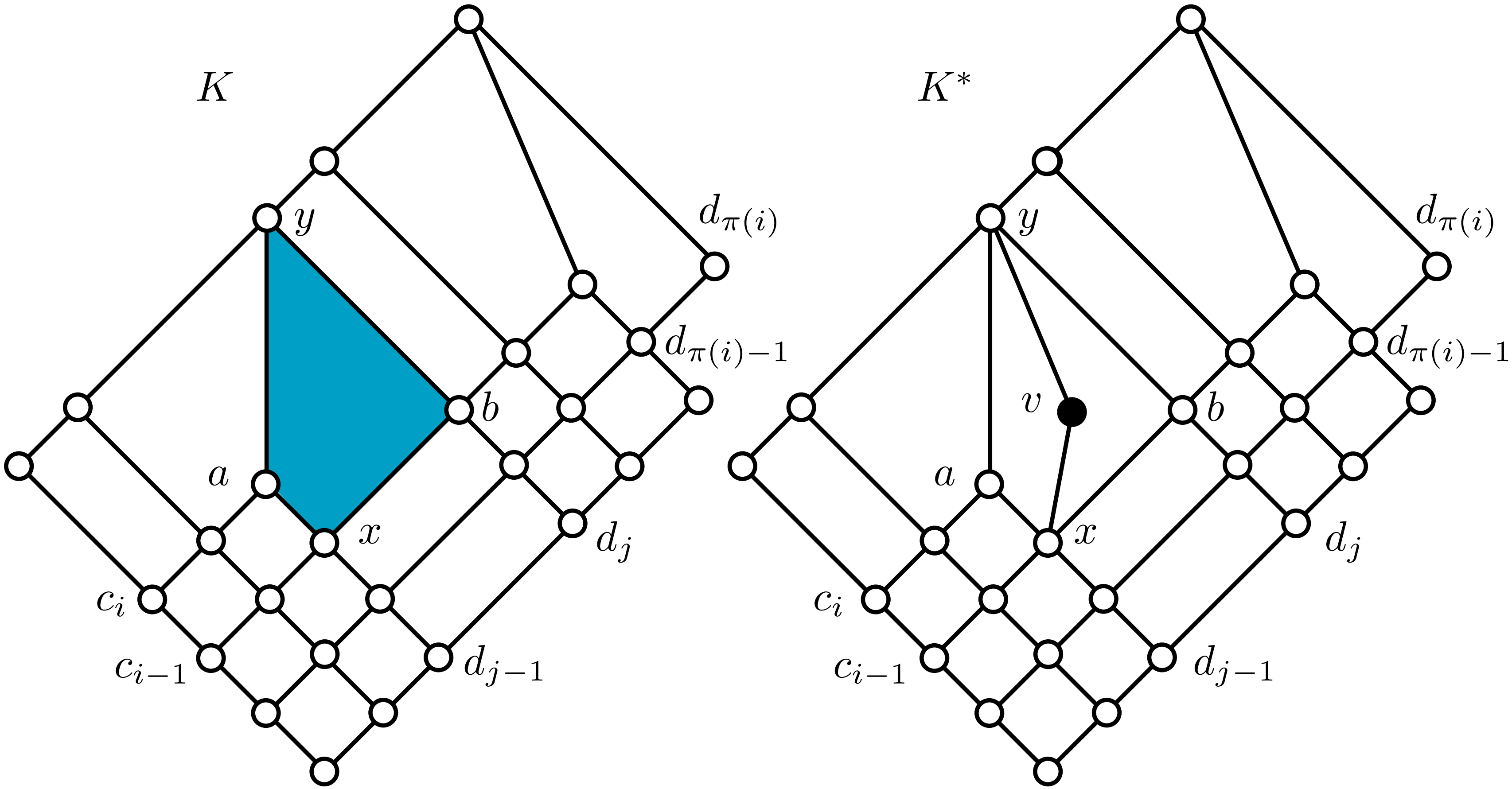




Denote the joins of c_{i-1} and c_i by d_{j-1}, d_j by x, y, a, b. The situation in L implies (very easy exercise) that $a \neq x \neq b$. Hence $\{x, a, b, y\}$ is a covering square by semimodularity.



(Something unusual:) Insert a new element v into K; we get K^*.

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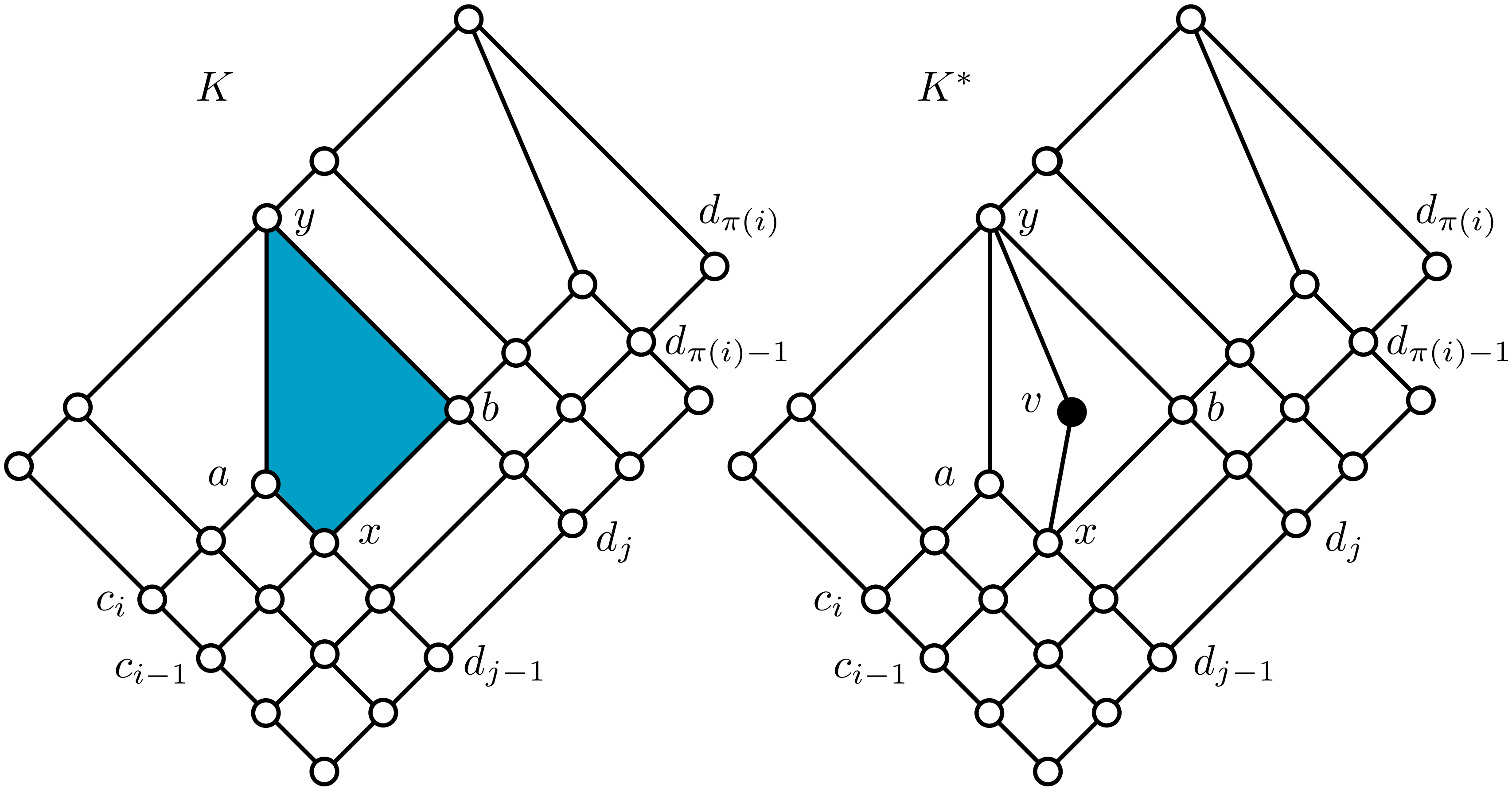
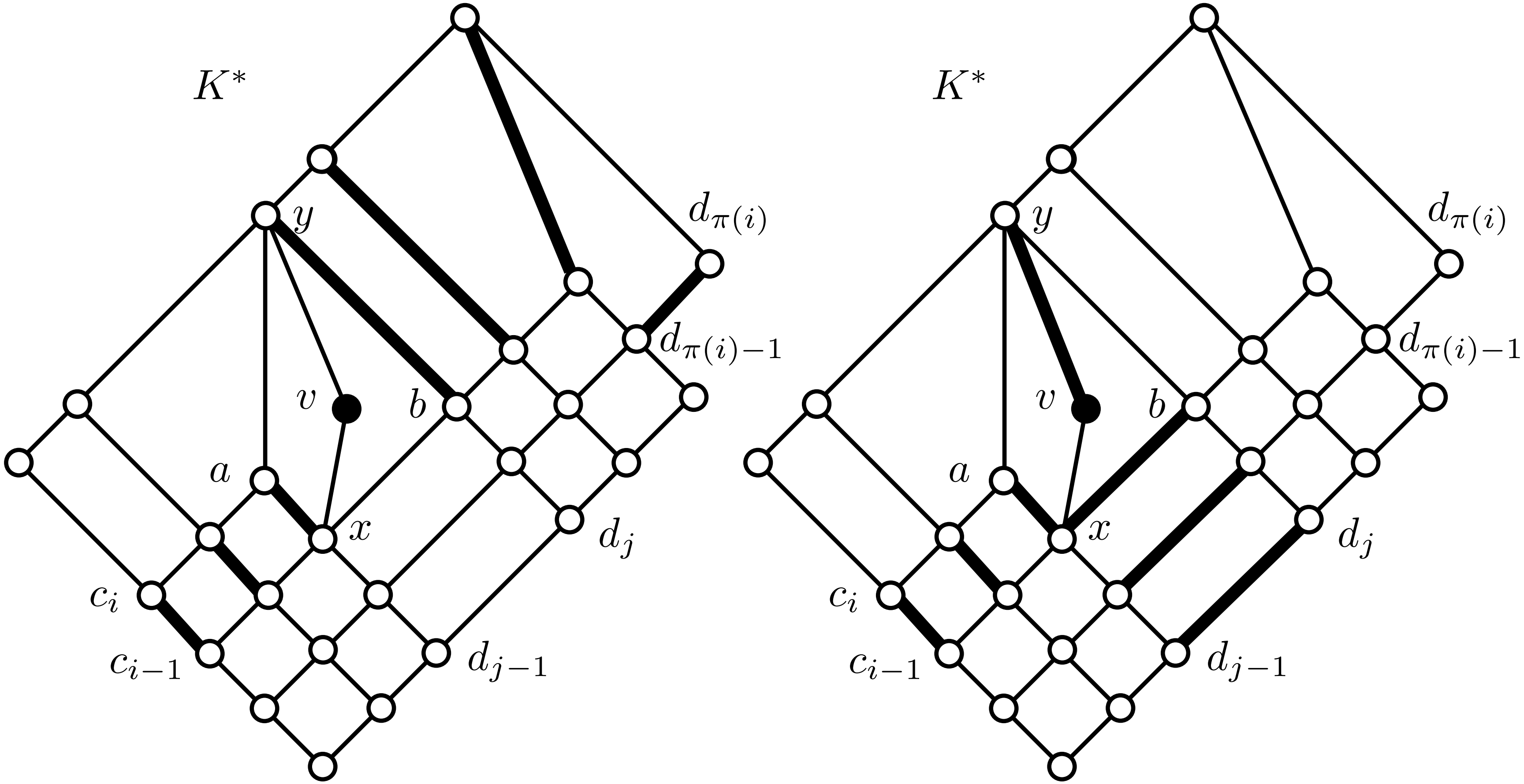
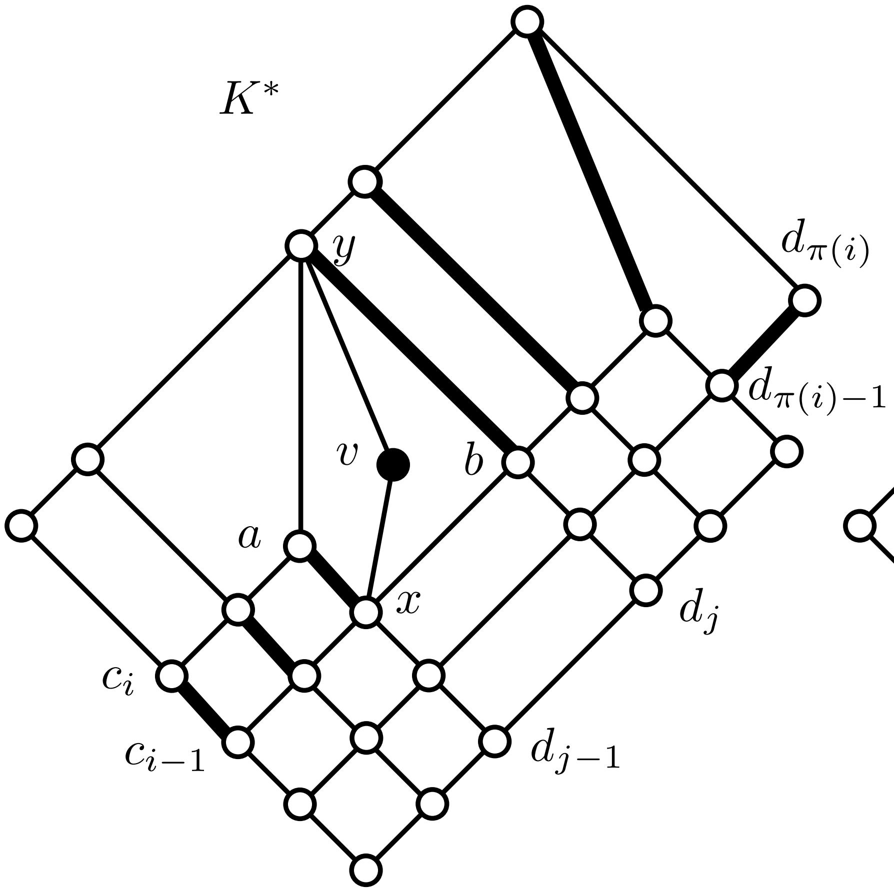
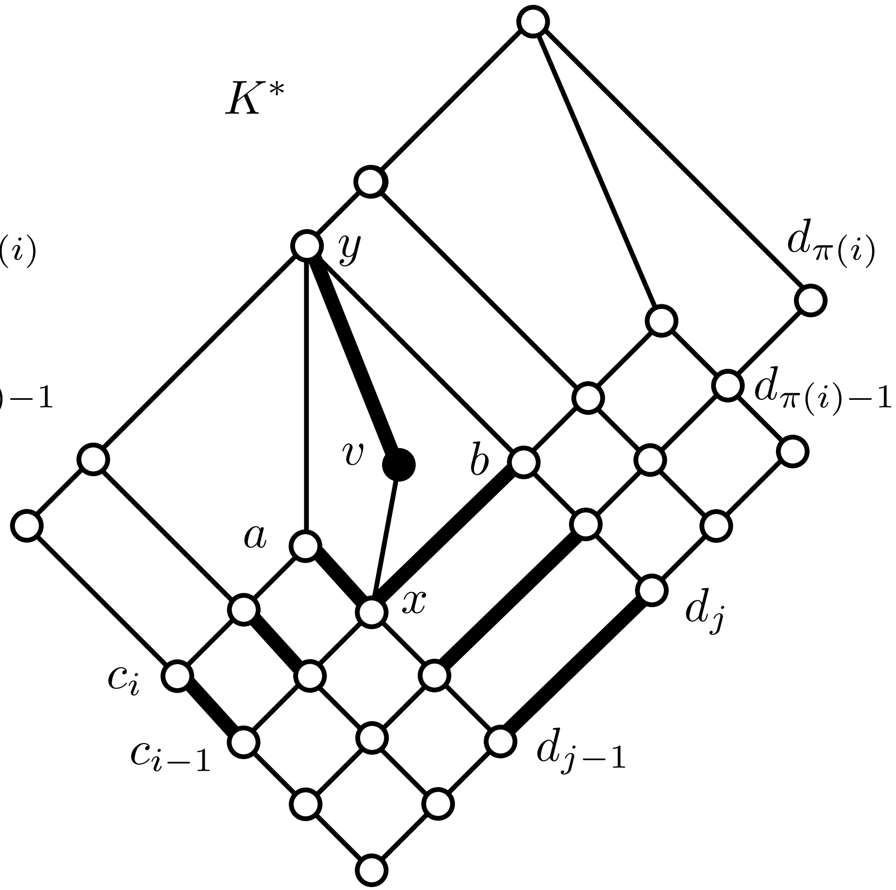


(Something unusual:) Insert a new element v into K; we get K^*. Note that v is not in L and K^* is **not a sublattice** of L, not even a join-subsemilattice of L, in general.



While the trajectories of K never ramify, the new element permits **exactly one** ramification (at v).

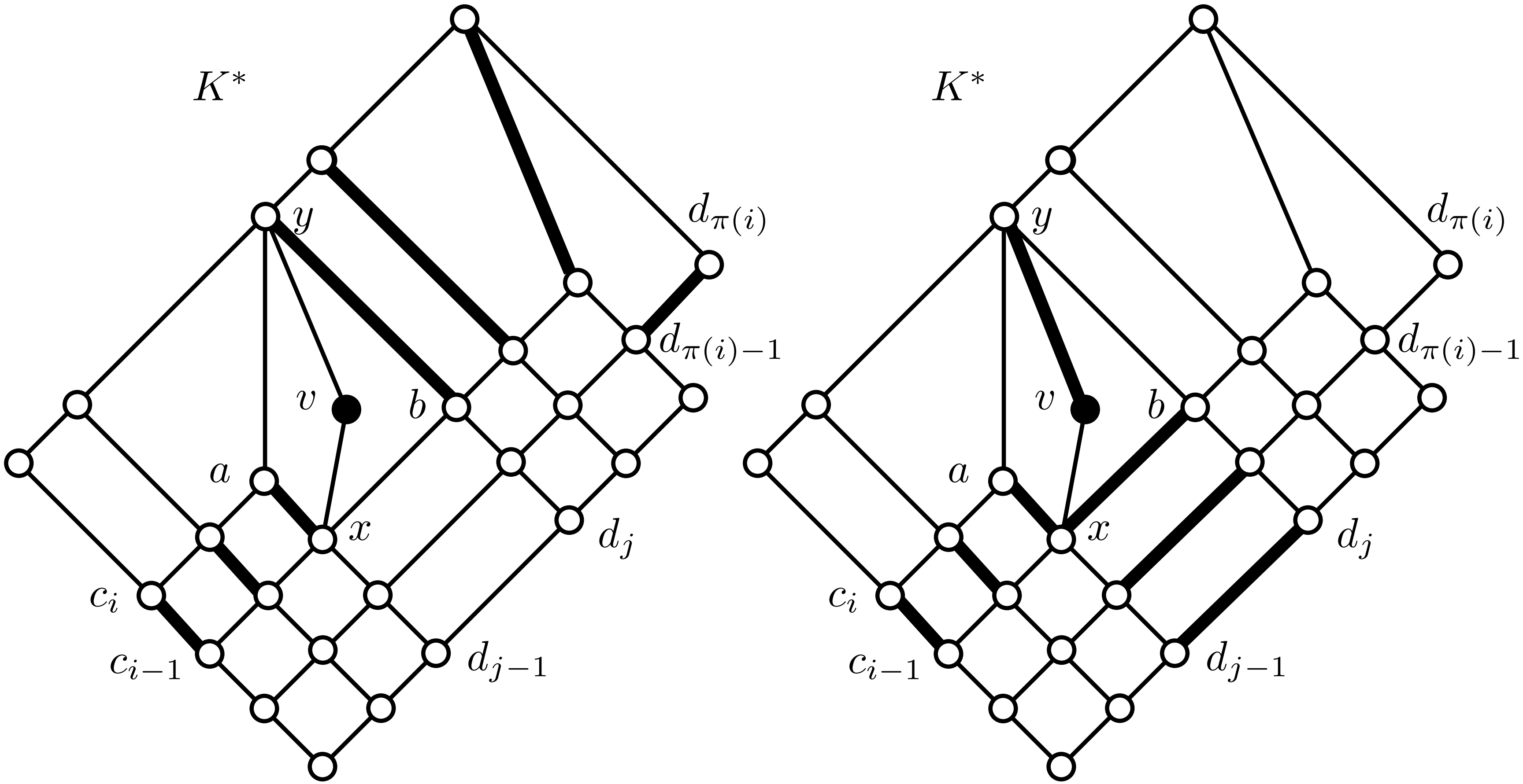




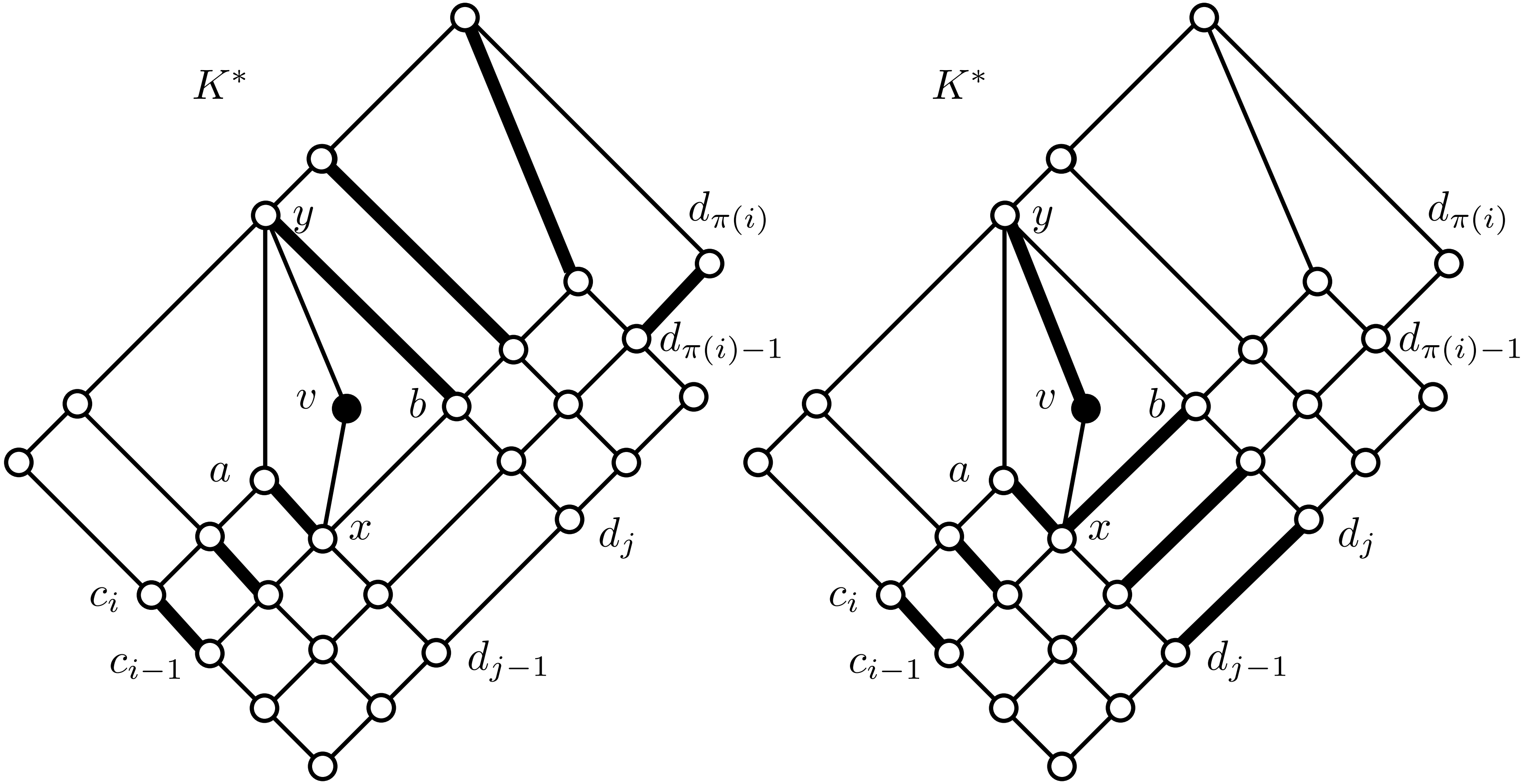
The old trajectory (on the left) pays no attention to v. It **keeps going straight** to the northeast for a while, then it may turn to the southeast, and arrives at the right (eastern) border at $[d_{\pi(i)-1}, d_{\pi(i)}]$.

The new one turns to the SE sooner

Czédli-Schmidt, 2010 $17'/3'$



The new trajectory (on the right) **turns** to the southeast **much sooner**; namely, already at v. Since it continues in K, it cannot turn to the northeast later. So, from v to the right boundary, it goes to the southeast, and finally stops at $[d_{j-1}, d_j]$



Since the new trajectory turns to the southeast sooner than the old one, it reaches the right boundary **lower** than the old one. Hence $j < \pi(i)$,as desired. Q.e.d.

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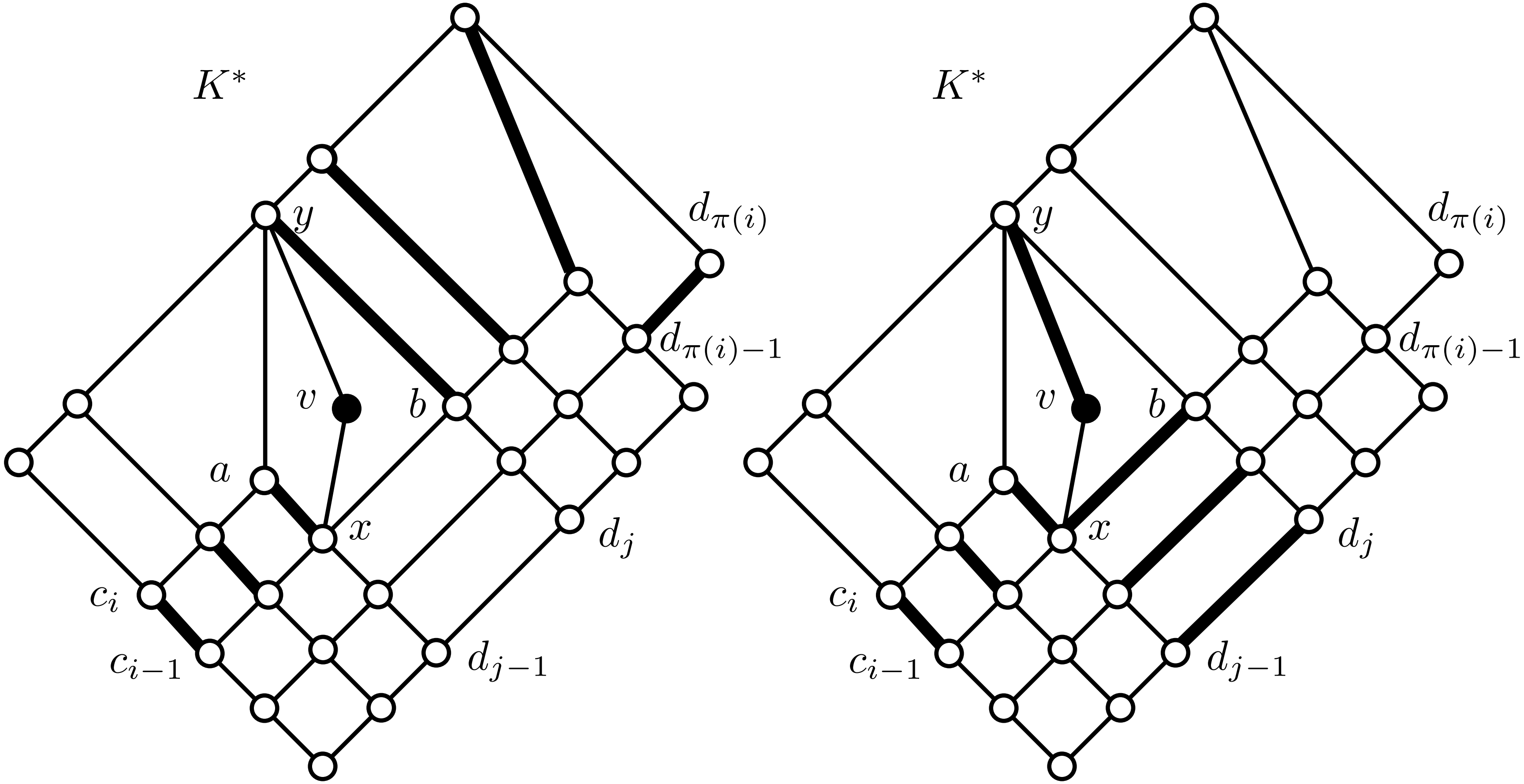

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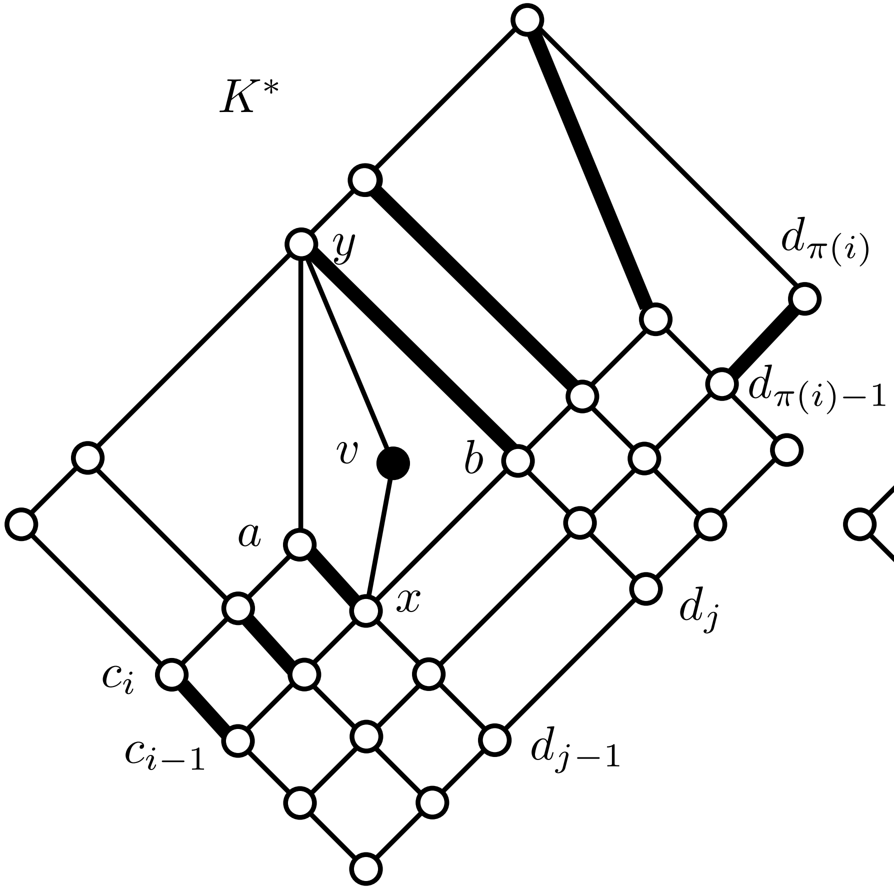

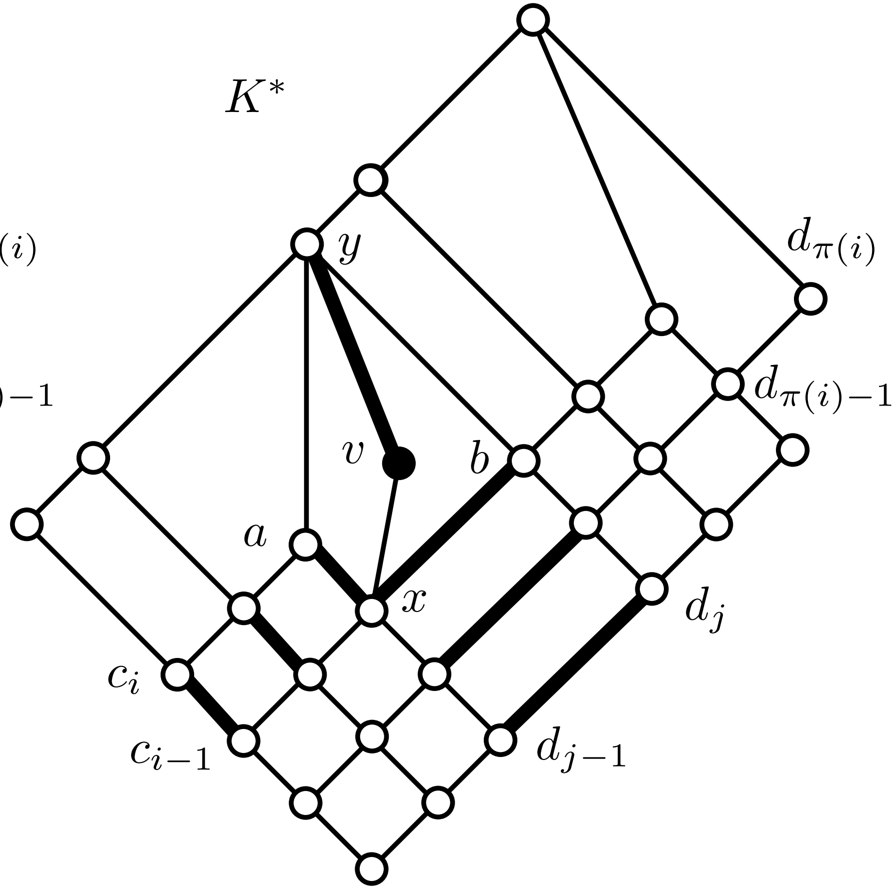

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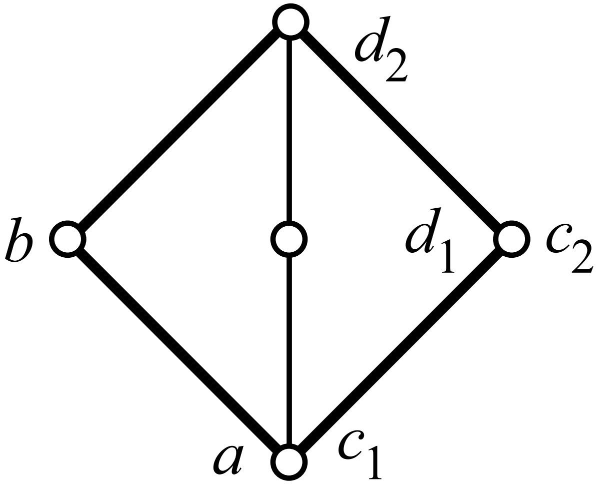
For a single prime interval $[c_{i-1}, c_i] = [a, b]$, there is no uniqueness!



$[a, b] \nearrow\searrow [c_j, d_j]$, for $j = 1, 2$.

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