Proper Classes associated to Grothendieck Categories

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- $R$-Mod is the category of left $R$-modules.
- $R$-simp is a complete irredundant set of representatives of the isomorphism classes of simple left $R$-modules.
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Definition

A **Proper Class** in $R$-mod is a family $\mathcal{E}$, of short exact sequences of left $R$-modules such that, if we denote by $E_m$ the monics of sequences of $\mathcal{E}$ and by $E_e$ the epics of sequences of $\mathcal{E}$, then the following conditions hold:

- **P0** $\mathcal{E}$ is closed under isomorphisms.
- **P1** All the splitting short exact sequences are in $\mathcal{E}$.
- **P2** If $\alpha, \beta \in E_m$ then $\alpha \beta \in E_m$ when the composition makes sense.
- **P2'** If $\alpha, \beta \in E_e$ then $\alpha \beta \in E_e$ when the composition makes sense.
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Definition

- A Proper Class **Injectively Generated** by a family of modules $\mathcal{O}$ is the greatest proper class $\iota^{-1}(\mathcal{O})$ such that each module in $\mathcal{O}$ is injective for all short exact sequence in $\iota^{-1}(\mathcal{O})$.

- A Proper Class **Coinjectively Generated** by a family of modules $\mathcal{O}$ is the least proper class $\mathcal{K}_i(\mathcal{O})$ such that all the short exact sequences $A \rightarrowtail B \twoheadrightarrow C$, where $C$ is in $\mathcal{O}$, are in $\mathcal{K}_i(\mathcal{O})$. 
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- **Injective Relative** If $\mathcal{E}$ is a class of short exact sequences. We say that a $\mathbb{R}$-module $M$ is injective relative to $\mathcal{E}$ if it is injective for all the short exact sequences in $\mathcal{E}$.

- **Coinjective Relative** A module $M$ is coinjective relative to $\mathcal{E}$ if all the short exact sequences that end in $M$ are in $\mathcal{E}$. 
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Notation: We use SEC for denote the family of all the short exact sequences in $R$-mod.
Lemma

Let $\mathcal{U} \subseteq \mathcal{V} \subseteq R – \text{Mod}$ and $\mathcal{D} \subseteq \mathcal{E} \subseteq \text{SEC}$

- $K^{-1}_i(\mathcal{U}) \subseteq K^{-1}_i(\mathcal{V})$
- $K_i(\mathcal{D}) \subseteq K_i(\mathcal{E})$

If $\mathcal{E}$ is a proper class, then $K_i(\mathcal{E})$ is a class closed under extensions, proper submodules, finite direct sums.

If $\mathcal{E}$ is a proper class, then a $R$-module $M$ is $\mathcal{E}$ – coinjective if and only if $A$ is a proper subgroup of $I$, for $I$ some injective module.
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Example 1: Consider the category of Abelian Groups. Then
\[ H = \{ A \rightarrow B \rightarrow C \mid A \text{ is pure in } B \} \] is a proper class and
and
- \( K_p(H) \)
- \( K_i(H) = \text{Pure subgroups of divisibles = divisibles} \)
- \( H = \iota^{-1}(\text{Cocyclics}) \)
  where Cocyclics are the groups \( \mathbb{Z}_{p^n} \) p is prime
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Definition

A Torsion Theory is a couple $\tau = (\mathcal{I}_\tau, \mathcal{F}_\tau)$ of classes of modules such that:

- $\mathcal{I}_\tau \cap \mathcal{F}_\tau = 0$
- $\mathcal{I}_\tau$ is closed under quotients
- $\mathcal{F}_\tau$ is closed under submodules
- For each module $D$, there exist $A \in \mathcal{I}_\tau$ and $C \in \mathcal{F}_\tau$ such that $A \rightarrow D \rightarrow C$ is exact.
Definition

An hereditary torsion theory is a torsion theory such that $\mathcal{I}_T$ is closed under submodules.

In the following we consider torsion theory instead of hereditary torsion theory unless otherwise stated. We denote as $R$-tors the family of the hereditary torsion theories.
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In the following we consider torsion theory instead of hereditary torsion theory unless otherwise stated. We denote as $R$-tors the family of the hereditary torsion theories.
**Definition**

If $\tau \in R - tors$ we say that an $R$-module $M$ is $\tau$-injective if it is injective for all the short exact sequences

$$\{A \hookrightarrow B \twoheadrightarrow C \mid C \in \mathcal{I}_\tau\}$$

**Definition**

If $\tau \in R - tors$ we say that an $R$-module $M$ is $\tau$-projective if it is projective for all the short exact sequences

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If $\tau \in R - \text{tors}$ we say that a $R$-module $M$ is $\tau$-divisible if it is injective for all the short exact sequences

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Definition

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$\{ A \rightarrowtail B \twoheadrightarrow C \mid A \in \mathcal{F}_\tau \}$
Theorem

(Walker) Let $\tau$ be a torsion theory and $D$ the family of short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ such that the induced sequence $A/T(A) \rightarrowtail B/T(B) \twoheadrightarrow C/T(C)$ is exact and splits, then $D = \iota^{-1}(F_\tau)$
Theorem

(Walker) Let $\tau$ be a torsion theory and $D$ the family of short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ such that the induced sequence $A/T(A) \rightarrowtail B/T(B) \twoheadrightarrow C/T(C)$ is exact and splits, then $D = \iota^{-1}(\mathcal{F}_\tau)$
**Theorem**

The left $R$-module $M$ is projective relative to the proper class $\iota^{-1}(\mathcal{F}_\tau)$ if and only if $\Ext^1(M, L) = 0$ for all $L$ in $\mathcal{I}_\tau$. 
Theorem

Let $\tau$ an hereditary torsion theory, then

$$\iota K_p^{-1}(\mathcal{I}_\tau) = \tau - \text{injectives}$$
Theorem

Let $\tau$ an hereditary torsion theory, then

$$K_i\pi^{-1}(I_\tau) = \tau - \text{injectives}$$
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Let $\tau$ an hereditary torsion theory, then

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Let $\tau$ an hereditary torsion theory, then

$$K_{p\tau}^{-1}(F_{\tau}) = \tau - \text{codivisbles}$$
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Let $\tau$ an hereditary torsion theory, then

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Theorem

Let $\tau$ an hereditary torsion theory, then

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Example 2
Consider the category of abelian groups and let $D$ the class of the divisible groups $R$ the class of the reduced groups. the Pair $(\mathcal{D}, \mathcal{R})$ is an hereditary torsion theory and we have the following:
Example 3

**Definition**

Let \( E : A \to B \to C \) a short exact sequence and \( h_n : C \to C \) such that \( h_n(x) = nx \) \( n \in \mathbb{Z} \). We say that \( E \) is quasi-pure if \( Eh_n \) is pure for some \( n \in \mathbb{Z} \).

All the short exact sequences quasi-pure form a proper class.
Now, if we consider the proper classes

\[ K^{-1}_i \subseteq \pi^{-1}(\tau - \text{codivisibles}) \]

we observe that

\[ \pi K^{-1}_i = \pi \pi^{-1}(\tau - \text{codivisibles}) = \tau - \text{codivisibles} \]

We also consider the concept of cover $\tau$-projective
Some Examples
Theorem

- $M$ is $\iota^{-1}(\mathcal{F}_\tau)$-projective if and only if $M$ is a direct summand of a direct sum of projective and torsion modules.
- $M$ is $\iota^{-1}(\mathcal{F}_\tau)$-coprojective if and only if $M$ is $\tau$-codivisible.
Sklyarenko, E.G. *Relative homological algebra in categories of modules*. Russian Mathematical Surveys, 33 No. 3, 97-137
(1978).

