Proper Classes associated to Grothendieck Categories

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Notation

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- R-Mod is the category of left R-modules.
- R-simp is a complete irredundant set of representatives of the isomorphism classes of simple left R-modules.

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Definition

A Proper Class in R-mod is a family $\mathcal{E}$, of short exact sequences of left R-modules such that, if we denote by $\mathcal{E}_m$ the monics of sequences of $\mathcal{E}$ and by $\mathcal{E}_e$ the epics of sequences of $\mathcal{E}$, then the following conditions hold:

- P0 $\mathcal{E}$ is closed under isomophisms.
- P1 All the splitting short exact sequences are in $\mathcal{E}$
- P2 If $\alpha, \beta \in \mathcal{E}_m$ then $\alpha\beta \in \mathcal{E}_m$ when the composition makes sense.
- P2' If $\alpha, \beta \in \mathcal{E}_e$ then $\alpha\beta \in \mathcal{E}_e$ when the composition makes sense.
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Definition

- A Proper Class Injectively Generated by a family of modules $\mathcal{O}$ is the greatest proper class $\iota^{-1}(\mathcal{O})$ such that each module in $\mathcal{O}$ is injective for all short exact sequence in $\iota^{-1}(\mathcal{O})$
- A Proper Class Coinjectively Generated by a family of modules $\mathcal{O}$ is the least proper class $\mathcal{K}_i(\mathcal{O})$ such that all the short exact sequences $A \rightarrowtail B \twoheadrightarrow C$, where C is in $\mathcal{O}$, are in $\mathcal{K}_i(\mathcal{O})$.

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Definition

- **Injective Relative** If $\mathcal{E}$ is a class of short exact sequences. We say that a R-module M is injective relative to $\mathcal{E}$ if it is injective for all the short exact sequences in $\mathcal{E}$.
- **Coinjective Relative** A module M is coinjective relative to $\mathcal{E}$ if all the short exact sequences that end in M are in $\mathcal{E}$

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Notation: We use SEC for denote the family of all the short exact sequences in R-mod.

Lemma

Let $\mathcal{U} \subseteq \mathcal{V} \subseteq R - Mod$ and $\mathcal{D} \subseteq \mathcal{E} \subseteq SEC$

- $K_i^{-1}(\mathcal{U}) \subset K_i^{-1}(\mathcal{V})$
- $K_i(\mathcal{D}) \subset K_i(\mathcal{E})$
- *If $\mathcal{E}$ is a proper class, then $K_i(\mathcal{E})$ is a class closed under extensions, proper submodules, finite direct sums.*
- *If $\mathcal{E}$ is a proper class, then a R-module M is $\mathcal{E}$ − coinjective if and only if A is a proper subgroup of I, for I some injective module.*

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Example 1: Consider the category of Abelian Groups. Then $H = \{A \rightarrowtail B \twoheadrightarrow C \mid A \text{ is pure in } B\}$ is a proper class and and

- $K_p(H)$
- $K_i(H)$=Pure subgroups of divisibles=divisibles
- $H = \iota^{-1}(Cocyclics)$
  where Cocyclics are the groups $\mathbb{Z}_{p^n}$ p is prime
- $H = \pi^{-1}(cyclics)$
- H has enough projectives and injectives

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Definition

A Torsion Theory is a couple $\tau = (\mathcal{T}_\tau, \mathcal{F}_\tau)$ of classes of modules such that:

- $\mathcal{T}_\tau \cap \mathcal{F}_\tau = 0$
- $\mathcal{T}_\tau$ is closed under quotients
- $\mathcal{F}_\tau$ is closed under submodules
- For each module D, there exist $A \in \mathcal{T}_\tau$ and $C \in \mathcal{F}_\tau$ such that $A \rightarrowtail D \twoheadrightarrow C$ is exact.

Definition

An hereditary torsion theory is a torsion theory such that $\mathcal{T}_\tau$ is closed under submodules.

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We denote as R-tors the family of the hereditary torsion theories.

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If $\tau \in R-tors$ we say that an R-module M is τ-injective if it is injective for all the short exact sequences

$$\{A \rightarrowtail B \twoheadrightarrow C \mid C \in \mathcal{T}_\tau\}$$

Definition

If $\tau \in R-tors$ we say that an R-module M is τ-projective if it is projective for all the short exact sequences

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Theorem

(Walker) Let τ *be a torsion theory and* $\mathcal{D}$ *the family of short exact sequences* $A \rightarrowtail B \twoheadrightarrow C$ *such that the induced sequence* $A/T(A) \rightarrowtail B/T(B) \twoheadrightarrow C/T(C)$ *is exact and splits, then* $\mathcal{D} = \iota^{-1}(\mathcal{F}_\tau)$

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Theorem

The left R-module M is projective relative to the proper class $\iota^{-1}(\mathcal{F}_\tau)$ if and only if $Ext^1(M, L) = 0$ for all L in $\mathcal{T}_\tau$

Theorem

Let τ an hereditary torsion theory, then

$$\iota K_p^{-1}(\mathcal{T}_\tau) = \tau - \textit{injectives}$$

Theorem

Let τ an hereditary torsion theory, then

$$K_i\pi^{-1}(\mathcal{T}_\tau) = \tau - injectives$$

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Let τ an hereditary torsion theory, then

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Example 2
Consider the category of abelian groups and let D the class of the divisible groups R the class of the reduced groups.
the Pair $(\mathcal{D}, \mathcal{R})$ is an hereditary torsion theory and we have the following:

Example 3

Definition

Let $E : A \rightarrowtail B \twoheadrightarrow C$ a short exact sequence and $h_n : C \longrightarrow C$ such that $h_n(x) = nx \;\; n \in \mathbb{Z}$. We say that E is quasi-pure if Eh_n is pure for some $n \in \mathbb{Z}$

All the short exact sequences quasi-pure form a proper class.

Now, if we consider the proper classes

$$K_i^{-1} \subseteq \pi^{-1}(\tau - \text{codivisibles})$$

we observe that

$$\pi K_i^{-1} = \pi\pi^{-1}(\tau - \text{codivisibles}) = \tau - \text{codivisibles}$$

We also consider the concept of cover τ-projective

Some Examples

Theorem

- *M is $\iota^{-1}(\mathcal{F}_\tau)$-projective if and only if M is a direct summand of a direct sum of projective and torsion modules.*
- *M es $\iota^{-1}(\mathcal{F}_\tau)$-coprojective if and only if M es τ-codivisible.*

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