Enveloping algebra for simple Malcev algebras

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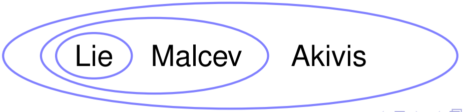
- A algebra finite dimensional over a field $\mathbb{K}$
- A^- algebra: replacing the product xy in A by the commutator $[x, y] = xy - yx$, $x, y \in A$.
- $Ak(A)$ algebra: provided with commutator $[x, y] = xy - yx$ and associator $\mathcal{A}(x, y, z) = (xy)z - x(yz)$, $x, y, z \in A$.

A		
associative	$\Longrightarrow$	A^- Lie algebra
alternative $\left[\begin{array}{l} x(xy) = x^2 y \\ (yx)x = yx^2, \quad x, y \in A \end{array}\right]$	$\Longrightarrow$	A^- Malcev algebra
not necessarily associative	$\Longrightarrow$	$Ak(A)$ Akivis algebra



CONVERSELY

Theorem (Poincaré-Birkhoff-Witt)

Any Lie algebra is isomorphic to a subalgebra of A^- for a suitable associative algebra A.

Theorem

An arbitrary Akivis algebra can be isomorphically embedded into an Akivis algebra $Ak(A)$ for an algebra A.

Reference: I. P. Shestakov, Every Akivis algebra is linear, *Geometriae Dedicata* 77 (1999), 215–223.

QUESTION:

Is any Malcev algebra isomorphic to a subalgebra of A^-, for some alternative algebra A?

- J. M. Pérez-Izquierdo and I. Shestakov: presented enveloping algebra of Malcev algebras (constructed in a more general way)

enveloping algebra	generalize the enveloping algebra of Lie algebra not alternative, in general has a basis of P-B-W Theorem type inherits properties of the enveloping of Lie algebras

Reference: J. M. Pérez-Izquierdo and I. P. Shestakov, An envelope for Malcev algebras, *J. Algebra* 272 (2004), 379–393.

Not alternative!

QUESTION: Is the enveloping algebra for simple Malcev algebras alternative?

ALGEBRAS

1. v s L endowed with a bilinear multiplication is *Lie algebra* if
   - L1 $[x,y] = -[y,x]$ (anti-symmetry)
   - L2 $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ (Jacobi identity)
2. v s M endowed with a bilinear multiplication is *Malcev algebra* if
   - M1 $xy = -yx$ (anti-symmetry)
   - M2 $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$ (Malcev identity)
3. v s A endowed with a bilinear multiplication and a trilinear multiplication $\mathcal{A}(\,,\,,\,)$ is *Akivis algebra* if
   - A1 $[x,y] = -[y,x]$ (anti-symmetry)
   - A2 $[[x,y],z] + [[y,z],x] + [[z,x],y] = \mathcal{A}(x,y,z) + \mathcal{A}(y,z,x) + \mathcal{A}(z,x,y)$ $-\mathcal{A}(y,x,z) - \mathcal{A}(x,z,y) - \mathcal{A}(z,y,x)$

SIMPLE MALCEV ALGEBRAS

M is *simple* if it has no ideals except itself and zero, and $MM \neq \{0\}$.

A simple Malcev algebra is either a simple Lie algebra or isomorphic to the 7-dim simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$.

Simple Malcev algebra: (if field $\mathbb{K}$ char$\neq 2, 3$)

$$\begin{cases} \text{simple Lie algebras} \\ \\ \text{isomorphic to } 7\text{-dim simple (non-Lie) Malcev algebra } M(\alpha, \beta, \gamma), \\ \text{with } \alpha\beta\gamma \neq 0 \end{cases}$$

SIMPLE MALCEV ALGEBRAS

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Simple Malcev algebra: (if field $\mathbb{K}$ algebraically closed, char$= 0$)

$$\begin{cases} \text{simple Lie algebras :} & \begin{array}{|l} \text{special linear algebra } A_n \ (\mathfrak{sl}(n+1,\mathbb{K}),\ n \geq 1) \\ \text{orthogonal algebra} B_n \ (\mathfrak{so}(2n+1,\mathbb{K}),\ n \geq 2) \\ \text{symplectic algebra} C_n \ (\mathfrak{sp}(2n,\mathbb{K}),\ n \geq 3) \\ \text{orthogonal algebra} D_n \ (\mathfrak{so}(2n,\mathbb{K}),\ n \geq 4) \\ \text{exceptional Lie algebras: } E_6,\ E_7, E_8,\ F_4,\ G_2 \end{array} \\ \text{isomorphic to } 7\text{-dim simple (non-Lie) Malcev algebra } M(-1,-1,-1) & \end{cases}$$

SIMPLE (NON-LIE) MALCEV ALGEBRA $M(\alpha,\beta,\gamma)$

- Each algebra $M(\alpha,\beta,\gamma)$ over a field $\mathbb{K}$ (char$\neq 2$) is isomorphic to the algebra $C^{-}/\mathbb{K}$, C is suitable Cayley-Dickson algebra over $\mathbb{K}$.
- Two algebras of this type are isomorphic $\Longleftrightarrow$ the corresponding Cayley-Dickson algebras are isomorphic.

If $\{e_1,\ldots,e_7\}$ is basis of $M(\alpha,\beta,\gamma)$, the multiplication table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	$-e_3$	0	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	αe_2	$-\beta e_1$	0	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	γe_1	γe_2	γe_3
e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	0	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	0	$-\beta\gamma e_1$
e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	0

ENVELOPING ALGEBRA OF A LIE ALGEBRA

▷ L Lie algebra.

The *universal enveloping algebra* of L is a pair $(\mathfrak{U}, \iota)$,

- $\mathfrak{U}$ is associative algebra with identity element 1
- $\iota : L \longrightarrow \mathfrak{U}^-$ is Lie homomorphism

such that for any associative algebra $\mathfrak{B}$ having an identity element 1 and any Lie homomorphism $\varphi : L \longrightarrow \mathfrak{B}^-$

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathfrak{B} \\ \mathfrak{U} & \end{tikzcd}$$

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there exists a unique homomorphism of algebras $\varphi' : \mathfrak{U} \longrightarrow \mathfrak{B}$ such that $\varphi'(1) = 1$ and $\varphi = \varphi' \circ \iota$

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathfrak{B} \\ \mathfrak{U} \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

PROPERTIES OF A ENVELOPING ALGEBRA

1. The pair $(\mathfrak{U}, \iota)$ is unique (up to an isomorphism).
2. $\mathfrak{U}$ is generated by the image $\iota(L)$ (as an algebra).
3. L_1, L_2 Lie algebras and $(\mathfrak{U}_1, \iota_1)$, $(\mathfrak{U}_2, \iota_2)$ are the respective universal enveloping algebras, homomorphism $\alpha : L_1 \longrightarrow L_2$. Then there exists a unique homomorphism $\alpha' : \mathfrak{U}_1 \longrightarrow \mathfrak{U}_2$ such that $\iota_2 \circ \alpha = \alpha' \circ \iota_1$,

$$\begin{tikzcd} L_1 \arrow[r, "\alpha"] \arrow[d, "\iota_1"'] & L_2 \arrow[d, "\iota_2"] \\ \mathfrak{U}_1 \arrow[r, "\alpha'"'] & \mathfrak{U}_2 \end{tikzcd}$$

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$$\begin{tikzcd} L_1 \arrow[r, "\alpha"] \arrow[d, "\iota_1"'] & L_2 \arrow[d, "\iota_2"] \\ \mathfrak{U}_1 \arrow[r, "\alpha'"'] & \mathfrak{U}_2 \end{tikzcd}$$

Consequence of property 3. instead of study representations of Lie algebras, we can study the representations of the universal enveloping algebra (associative)

4. I bilateral ideal in L and $\mathfrak{I}$ ideal in $\mathfrak{U}$ generated by $\iota(I)$. If $l \in L$ then $\jmath : l + I \longrightarrow \iota(l) + \mathfrak{I}$ is a homomorphism of L/I into $\mathfrak{B}^-$, where $\mathfrak{B} = \mathfrak{U}/\mathfrak{I}$, and $(\mathfrak{B}, \jmath)$ is a universal enveloping algebra for L/I.
5. $\mathfrak{U}$ has unique anti-automorphism π such that $\pi \circ \iota = -\iota$ and $\pi^2 = 1$.
6. There is unique homomorphism δ of $\mathfrak{U}$ into $\mathfrak{U} \otimes \mathfrak{U}$ (the *diagonal mapping* of $\mathfrak{U}$) such that $\delta(\iota(a)) = \iota(a) \otimes 1 + 1 \otimes \iota(a)$, $a \in L$.
7. If D is a derivation in L then there exists unique derivation D' in $\mathfrak{U}$ such that $\iota \circ D = D' \circ \iota$

$$\begin{tikzcd} L \arrow[r, "D"] \arrow[d, "\iota"'] & L \arrow[d, "\iota"] \\ \mathfrak{U} \arrow[r, "D'"'] & \mathfrak{U} \end{tikzcd}$$

EXISTENCE OF ENVELOPING ALGEBRA OF A LIE ALGEBRA

- Tensor algebra of Lie algebra L:

$$T(L) = \bigcup_{i=0}^{\infty} T^i L = \underbrace{T^0 L}_{\mathbb{K}} \cup \underbrace{T^1 L}_{L} \cup \underbrace{T^2 L}_{L \otimes L} \cup \cdots \cup \underbrace{T^m L}_{\substack{L\otimes\cdots\otimes L \\ \text{(m copies)}}} \cup \cdots$$

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m \in T^{m+k} L$$

Tensor algebra $T(L)$	associative algebra with unit element generated by 1 with any basis of L

- J two sided ideal in $T(L)$ generated by the elements:

$$x \otimes y - y \otimes x - [x, y], \forall\, x, y \in L.$$

- $\mathfrak{U}(L) = T(L)/J$ and $\iota : L \longrightarrow \mathfrak{U}(L)$ defined by $\iota(x) = x + J$, $x \in L$

$$\boxed{(\mathfrak{U}(L), \iota) \text{ enveloping algebra of } L}$$

Poincaré-Birkhoff-Witt Theorem

L Lie algebra (finite or infinite dimensional)
$\{x_1, x_2, \ldots\}$ ordered basis of L. Then:

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}, \; i_1 \leq i_2 \leq \cdots \leq i_k, \; k \in \mathbb{N}.$$

with the unit element, form a basis of $\mathfrak{U}(L)$.

If $\dim\ L < \infty$
$\{x_1, \ldots, x_n\}$ ordered basis of L. Then:

$$x_1^{m_1} \otimes \cdots \otimes x_n^{m_n}, \text{ with } m_i \geq 0 \; (i = 1, \ldots, n)$$

form a basis of $\mathfrak{U}(L)$.

OUR APPROACH:

- Root space decomposition of (non-Lie) Malcev algebra $M(\alpha,\beta,\gamma)$:

$$M(\alpha,\beta,\gamma) = \overbrace{H}^{<h=e_1>} \oplus M_{i\sqrt{\alpha}} \oplus M_{-i\sqrt{\alpha}},$$

$$M_{i\sqrt{\alpha}} = \left\langle x_1 = e_3 + i\sqrt{\alpha}e_2, x_2 = e_5 + i\sqrt{\alpha}e_4, x_3 = e_7 - i\sqrt{\alpha}e_6 \right\rangle,$$
$$M_{-i\sqrt{\alpha}} = \left\langle y_1 = e_3 - i\sqrt{\alpha}e_2, y_2 = e_5 - i\sqrt{\alpha}e_4, y_3 = e_7 + i\sqrt{\alpha}e_6 \right\rangle.$$

multiplication table:

	h	x_1	x_2	x_3	y_1	y_2	y_3
h	0	$i\sqrt{\alpha}x_1$	$i\sqrt{\alpha}x_2$	$i\sqrt{\alpha}x_3$	$-i\sqrt{\alpha}y_1$	$-i\sqrt{\alpha}y_2$	$-i\sqrt{\alpha}y_3$
x_1	$-i\sqrt{\alpha}x_1$	0	$2i\sqrt{\alpha}y_3$	$-2i\sqrt{\alpha}\beta y_2$	$2i\sqrt{\alpha}\beta h$	0	0
x_2	$-i\sqrt{\alpha}x_2$	$-2i\sqrt{\alpha}y_3$	0	$2i\sqrt{\alpha}\gamma y_1$	0	$2i\sqrt{\alpha}\gamma h$	0
x_3	$-i\sqrt{\alpha}x_3$	$2i\sqrt{\alpha}\beta y_2$	$-2i\sqrt{\alpha}\gamma y_1$	0	0	0	$2i\sqrt{\alpha}\beta\gamma h$
y_1	$i\sqrt{\alpha}y_1$	$-2i\sqrt{\alpha}\beta h$	0	0	0	$-2i\sqrt{\alpha}x_3$	$2i\sqrt{\alpha}\beta x_2$
y_2	$i\sqrt{\alpha}y_2$	0	$-2i\sqrt{\alpha}\gamma h$	0	$2i\sqrt{\alpha}x_3$	0	$-2i\sqrt{\alpha}\gamma x_1$
y_3	$i\sqrt{\alpha}y_3$	0	0	$-2i\sqrt{\alpha}\beta\gamma h$	$-2i\sqrt{\alpha}\beta x_2$	$2i\sqrt{\alpha}\gamma x_1$	0

OUR AIM:

Recover the initial algebra of the simple (non-Lie) Malcev algebra 7-dimensional $M(\alpha, \beta, \gamma)$.

Enveloping algebra $U(M(\alpha, \beta, \gamma))$:

- alternative algebra generated by $h, x_1, x_2, x_3, y_1, y_2, y_3$
- multiplication xy satisfying relation: $xy - yx = [x, y]$, where commutator $[,]$ is multiplication in $M(\alpha, \beta, \gamma)$

Relations holding in any alternative algebra:

$$\boxed{(x,y,z) = \frac{1}{6}J(x,y,z), \quad [x,y]\circ(x,y,z) = 0,} \quad \forall\, x,y,z \in U(M(\alpha,\beta,\gamma))$$

Jacobian $J(x,y,z) = [[x,y],z] + [[y,z],x] + [[z,x],y]$, associator $(x,y,z) = (xy)z - x(yz)$, Jordan product $x \circ y = xy + yx$.

Reference: K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov, A. I. Shirshov, *Rings that are nearly associative* Academic Press, New York, 1982.

Multiplication table in $U(M(\alpha,\beta,\gamma))$:

	h	x_1	x_2	x_3	y_1	y_2	y_3
h	0	$i\frac{\sqrt{\alpha}}{2}x_1$	$i\frac{\sqrt{\alpha}}{2}x_2$	$i\frac{\sqrt{\alpha}}{2}x_3$	$-i\frac{\sqrt{\alpha}}{2}y_1$	$-i\frac{\sqrt{\alpha}}{2}y_2$	$-i\frac{\sqrt{\alpha}}{2}y_3$
x_1	$-i\frac{\sqrt{\alpha}}{2}x_1$	0	$i\sqrt{\alpha}y_3$	$-i\sqrt{\alpha}\beta y_2$	$i\sqrt{\alpha}\beta h$	0	0
x_2	$-i\frac{\sqrt{\alpha}}{2}x_2$	$-i\sqrt{\alpha}y_3$	0	$i\sqrt{\alpha}\gamma y_1$	0	$i\sqrt{\alpha}\gamma h$	0
x_3	$-i\frac{\sqrt{\alpha}}{2}x_3$	$i\sqrt{\alpha}\beta y_2$	$-i\sqrt{\alpha}\gamma y_1$	0	0	0	$i\sqrt{\alpha}\beta\gamma h$
y_1	$i\frac{\sqrt{\alpha}}{2}y_1$	$-i\sqrt{\alpha}\beta h$	0	0	0	$-i\sqrt{\alpha}x_3$	$i\sqrt{\alpha}\beta x_2$
y_2	$i\frac{\sqrt{\alpha}}{2}y_2$	0	$-i\sqrt{\alpha}\gamma h$	0	$i\sqrt{\alpha}x_3$	0	$-i\sqrt{\alpha}\gamma x_1$
y_3	$i\frac{\sqrt{\alpha}}{2}y_3$	0	0	$-i\sqrt{\alpha}\beta\gamma h$	$-i\sqrt{\alpha}\beta x_2$	$i\sqrt{\alpha}\gamma x_1$	0

PROPOSITION

The algebra $U(M(\alpha, \beta, \gamma))$ is alternative $\iff$ it is trivial.

PROPOSITION

The algebra $U(M(\alpha, \beta, \gamma))$ is alternative $\Longleftrightarrow$ it is trivial.

CONCLUSION:

A associative $\Longrightarrow$ A^- Lie algebra

A alternative $\Longrightarrow$ A^- Malcev algebra

Given an simple Malcev algebra M of finite dimension:

• If M is Lie algebra then there exists an associative algebra A such that $M \subset A^-$

• If M is Malcev algebra then does not exist an alternative algebra A (apart from the trivial) such that $M \subset A^-$