Enveloping algebra for simple Malcev algebras

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(joint work with Sara Merino)

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\[ A \text{ algebra finite dimensional over a field } \mathbb{K} \]
\[ A^- \text{ algebra: replacing the product } xy \text{ in } A \text{ by the commutator } [x, y] = xy - yx, \ x, y \in A. \]
\[ Ak(A) \text{ algebra: provided with commutator } [x, y] = xy - yx \text{ and associator } A(x, y, z) = (xy)z - x(yz), \ x, y, z \in A. \]

\[ \begin{array}{c|c}
A & \text{associative} \\
\hline
\Rightarrow & A^- \text{ Lie algebra} \\
\hline
\text{alternative} & \Rightarrow A^- \text{ Malcev algebra} \\
\hline
\text{not necessarily associative} & \Rightarrow Ak(A) \text{ Akivis algebra}
\end{array} \]
Theorem (Poincaré-Birkhoff-Witt)

Any Lie algebra is isomorphic to a subalgebra of $A^-$ for a suitable associative algebra $A$.

Theorem

An arbitrary Akivis algebra can be isomorphically embedded into an Akivis algebra $A_k(A)$ for an algebra $A$.

QUESTION:

Is any Malcev algebra isomorphic to a subalgebra of $A^-$, for some alternative algebra $A$?

J. M. Pérez-Izquierdo and I. Shestakov: presented enveloping algebra of Malcev algebras (constructed in a more general way)

- Enveloping algebra generalize the enveloping algebra of Lie algebra
- Not alternative, in general
- Has a basis of P-B-W Theorem type
- Inherits properties of the enveloping of Lie algebras


QUESTION: Is the enveloping algebra for simple Malcev algebras alternative?
ALGEBRAS

1. Let $L$ endowed with a bilinear multiplication is **Lie algebra** if
   - $L_1 \ [x, y] = -[y, x]$ (anti-symmetry)
   - $L_2 \ [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity)

2. Let $M$ endowed with a bilinear multiplication is **Malcev algebra** if
   - $M_1 \ xy = -yx$ (anti-symmetry)
   - $M_2 \ (xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$ (Malcev identity)

3. Let $A$ endowed with a bilinear multiplication and a trilinear multiplication $A(\ ,\ )$ is **Akivis algebra** if
   - $A_1 \ [x, y] = -[y, x]$ (anti-symmetry)
   - $A_2 \ [[x, y], z] + [[y, z], x] + [[z, x], y] = A(x, y, z) + A(y, z, x) + A(z, x, y)
     \ - A(y, x, z) - A(x, z, y) - A(z, y, x)$
SIMPLE MALCEV ALGEBRAS

$M$ is \textit{simple} if it has no ideals except itself and zero, and $MM \neq \{0\}$.

A simple Malcev algebra is either a simple Lie algebra or isomorphic to the 7-dim simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$.

**Simple Malcev algebra:** (if field $K \text{ char} \neq 2, 3$)

- simple Lie algebras
- isomorphic to 7-dim simple (non-Lie) Malcev algebra $M(\alpha, \beta, \gamma)$, with $\alpha \beta \gamma \neq 0$
Motivation

Simple Malcev algebras

Enveloping algebra of Lie algebras

Enveloping algebra of simple Malcev algebras

SIMPLE MALCEV ALGEBRAS

$M$ is simple if it has no ideals except itself and zero, and $MM \neq \{0\}$.

A simple Malcev algebra is either a simple Lie algebra or isomorphic to the 7-dim simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$.

Simple Malcev algebra: (if field $K$ algebraically closed, char $= 0$)

- simple Lie algebras:
  - special linear algebra $A_n (\mathfrak{sl}(n + 1, K), n \geq 1)$
  - orthogonal algebra $B_n (\mathfrak{so}(2n + 1, K), n \geq 2)$
  - symplectic algebra $C_n (\mathfrak{sp}(2n, K), n \geq 3)$
  - orthogonal algebra $D_n (\mathfrak{so}(2n, K), n \geq 4)$
- exceptional Lie algebras: $E_6, E_7, E_8, F_4, G_2$

isomorphic to 7-dim simple (non-Lie) Malcev algebra $M(-1, -1, -1)$
SIMPLE (NON-LIE) MALCEV ALGEBRA $M(\alpha, \beta, \gamma)$

- Each algebra $M(\alpha, \beta, \gamma)$ over a field $\mathbb{K}$ (char $\neq 2$) is isomorphic to the algebra $C^-/\mathbb{K}$, $C$ is suitable Cayley-Dickson algebra over $\mathbb{K}$.
- Two algebras of this type are isomorphic $\iff$ the corresponding Cayley-Dickson algebras are isomorphic.

If $\{e_1, \ldots, e_7\}$ is basis of $M(\alpha, \beta, \gamma)$, the multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_5$</th>
<th>$e_6$</th>
<th>$e_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>0</td>
<td>$e_3$</td>
<td>$-\alpha e_2$</td>
<td>$e_5$</td>
<td>$-\alpha e_4$</td>
<td>$-e_7$</td>
<td>$\alpha e_6$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$-e_3$</td>
<td>0</td>
<td>$\beta e_1$</td>
<td>$e_6$</td>
<td>$e_7$</td>
<td>$-\beta e_4$</td>
<td>$-\beta e_5$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$\alpha e_2$</td>
<td>$-\beta e_1$</td>
<td>0</td>
<td>$e_7$</td>
<td>$-\alpha e_6$</td>
<td>$\beta e_5$</td>
<td>$-\alpha \beta e_4$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>$-e_5$</td>
<td>$-e_6$</td>
<td>$-e_7$</td>
<td>0</td>
<td>$\gamma e_1$</td>
<td>$\gamma e_2$</td>
<td>$\gamma e_3$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>$\alpha e_4$</td>
<td>$-e_7$</td>
<td>$\alpha e_6$</td>
<td>$-\gamma e_1$</td>
<td>0</td>
<td>$-\gamma e_3$</td>
<td>$\alpha \gamma e_2$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>$e_7$</td>
<td>$\beta e_4$</td>
<td>$-\beta e_5$</td>
<td>$-\gamma e_2$</td>
<td>$\gamma e_3$</td>
<td>0</td>
<td>$-\beta \gamma e_1$</td>
</tr>
<tr>
<td>$e_7$</td>
<td>$-\alpha e_6$</td>
<td>$\beta e_5$</td>
<td>$\alpha \beta e_4$</td>
<td>$-\gamma e_3$</td>
<td>$-\alpha \gamma e_2$</td>
<td>$\beta \gamma e_1$</td>
<td>0</td>
</tr>
</tbody>
</table>
The **universal enveloping algebra** of $L$ is a pair $(\mathcal{U}, \iota)$,

- $\mathcal{U}$ is associative algebra with identity element $1$
- $\iota : L \rightarrow \mathcal{U}^{-}$ is Lie homomorphism

such that for any associative algebra $\mathcal{B}$ having an identity element $1$ and any Lie homomorphism $\varphi : L \rightarrow \mathcal{B}^{-}$

\[
L \xrightarrow{\varphi} \mathcal{B} \\
\downarrow \iota \\
\mathcal{U}
\]
The **universal enveloping algebra** of $L$ is a pair $(\mathcal{U}, \iota)$,

- $\mathcal{U}$ is associative algebra with identity element $1$
- $\iota : L \rightarrow \mathcal{U}$ is Lie homomorphism

such that for any associative algebra $\mathcal{B}$ having an identity element $1$ and any Lie homomorphism $\varphi : L \rightarrow \mathcal{B}$, there exists a unique homomorphism of algebras $\varphi' : \mathcal{U} \rightarrow \mathcal{B}$ such that $\varphi'(1) = 1$ and $\varphi = \varphi' \circ \iota$. 

$$
\begin{array}{ccc}
L & \xrightarrow{\varphi} & \mathcal{B} \\
\downarrow \iota & & \downarrow \\
\mathcal{U} & \xrightarrow{\varphi'} & \\
\end{array}
$$
**PROPERTIES OF A ENVELOPING ALGEBRA**

1. The pair \((\mathcal{U}, \iota)\) is unique (up to an isomorphism).
2. \(\mathcal{U}\) is generated by the image \(\iota(L)\) (as an algebra).
3. \(L_1, L_2\) Lie algebras and \((\mathcal{U}_1, \iota_1), (\mathcal{U}_2, \iota_2)\) are the respective universal enveloping algebras, homomorphism \(\alpha : L_1 \rightarrow L_2\). Then there exists a unique homomorphism \(\alpha' : \mathcal{U}_1 \rightarrow \mathcal{U}_2\) such that \(\iota_2 \circ \alpha = \alpha' \circ \iota_1\),

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\alpha} & L_2 \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
\mathcal{U}_1 & \xrightarrow{\alpha'} & \mathcal{U}_2
\end{array}
\]
1. The pair \((\mathcal{U}, \iota)\) is unique (up to an isomorphism).

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3. \(L_1, L_2\) Lie algebras and \((\mathcal{U}_1, \iota_1)\), \((\mathcal{U}_2, \iota_2)\) are the respective universal enveloping algebras, homomorphism \(\alpha : L_1 \rightarrow L_2\). Then there exists a unique homomorphism \(\alpha' : \mathcal{U}_1 \rightarrow \mathcal{U}_2\) such that \(\iota_2 \circ \alpha = \alpha' \circ \iota_1\),

Consequence of property 3. instead of study representations of Lie algebras, we can study the representations of the universal enveloping algebra (associative).
4. Let $I$ be a bilateral ideal in $L$ and $\mathcal{I}$ ideal in $\mathcal{U}$ generated by $\iota(I)$. If $l \in L$ then $\eta : l + I \rightarrow \iota(l) + \mathcal{I}$ is a homomorphism of $L/I$ into $\mathcal{B}^-$, where $\mathcal{B} = \mathcal{U}/\mathcal{I}$, and $(\mathcal{B}, \eta)$ is a universal enveloping algebra for $L/I$.

5. $\mathcal{U}$ has unique anti-automorphism $\pi$ such that $\pi \circ \iota = -\iota$ and $\pi^2 = 1$.

6. There is unique homomorphism $\delta$ of $\mathcal{U}$ into $\mathcal{U} \otimes \mathcal{U}$ (the diagonal mapping of $\mathcal{U}$) such that $\delta(\iota(a)) = \iota(a) \otimes 1 + 1 \otimes \iota(a)$, $a \in L$.

7. If $D$ is a derivation in $L$ then there exists unique derivation $D'$ in $\mathcal{U}$ such that $\iota \circ D = D' \circ \iota$.

\[
\begin{array}{c}
L \xrightarrow{D} L \\
\downarrow \iota \quad \downarrow \iota \\
\mathcal{U} \xrightarrow{D'} \mathcal{U}
\end{array}
\]
**EXISTENCE OF ENVELOPING ALGEBRA OF A LIE ALGEBRA**

- **Tensor algebra of Lie algebra** $L$:

  $$T(L) = \bigcup_{i=0}^{\infty} T^i L = T^0_L \oplus T^1_L \oplus T^2_L \oplus \cdots \oplus T^m_L \oplus \cdots$$

  $$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m \in T^{m+k}L$$

  Tensor algebra $T(L)$ is an associative algebra with unit element generated by $1$ with any basis of $L$.

- **$J$ two sided ideal in $T(L)$** generated by the elements:

  $$x \otimes y - y \otimes x - [x, y], \forall x, y \in L.$$

- **$\mathfrak{U}(L) = T(L)/J$** and $\iota : L \rightarrow \mathfrak{U}(L)$ defined by $\iota(x) = x + J, x \in L$.

  $$(\mathfrak{U}(L), \iota)$$ is an enveloping algebra of $L$$
**Poincaré-Birkhoff-Witt Theorem**

$L$ Lie algebra (finite or infinite dimensional)  
\{x_1, x_2, \ldots\} ordered basis of $L$. Then:  

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}, \quad i_1 \leq i_2 \leq \cdots \leq i_k, \quad k \in \mathbb{N}.$$  

with the unit element, form a basis of $U(L)$.

If $\dim L < \infty$  
\{x_1, \ldots, x_n\} ordered basis of $L$. Then:  

$$x_1^{m_1} \otimes \cdots \otimes x_n^{m_n}, \quad \text{with } m_i \geq 0 \ (i = 1, \ldots, n)$$  

form a basis of $U(L)$. 

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Enveloping for simple Malcev algebras  
ICAL Prague 2010
 OUR APPROACH:

▶ Root space decomposition of (non-Lie) Malcev algebra $M(\alpha, \beta, \gamma)$:

$$M(\alpha, \beta, \gamma) = \langle h=e_1 \rangle \bigoplus H \bigoplus M_{i\sqrt{\alpha}} \bigoplus M_{-i\sqrt{\alpha}},$$

$$M_{i\sqrt{\alpha}} = \langle x_1 = e_3 + i\sqrt{\alpha}e_2, x_2 = e_5 + i\sqrt{\alpha}e_4, x_3 = e_7 - i\sqrt{\alpha}e_6 \rangle,$$

$$M_{-i\sqrt{\alpha}} = \langle y_1 = e_3 - i\sqrt{\alpha}e_2, y_2 = e_5 - i\sqrt{\alpha}e_4, y_3 = e_7 + i\sqrt{\alpha}e_6 \rangle.$$

multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>$h$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0</td>
<td>$i\sqrt{\alpha}x_1$</td>
<td>$i\sqrt{\alpha}x_2$</td>
<td>$i\sqrt{\alpha}x_3$</td>
<td>$-i\sqrt{\alpha}y_1$</td>
<td>$-i\sqrt{\alpha}y_2$</td>
<td>$-i\sqrt{\alpha}y_3$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$-i\sqrt{\alpha}x_1$</td>
<td>0</td>
<td>$2i\sqrt{\alpha}y_3$</td>
<td>$-2i\sqrt{\alpha}\beta y_2$</td>
<td>$2i\sqrt{\alpha}\beta h$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$-i\sqrt{\alpha}x_2$</td>
<td>$-2i\sqrt{\alpha}y_3$</td>
<td>0</td>
<td>$2i\sqrt{\alpha}\gamma y_1$</td>
<td>0</td>
<td>2i$\sqrt{\alpha}\gamma h$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$-i\sqrt{\alpha}x_3$</td>
<td>$2i\sqrt{\alpha}\beta y_2$</td>
<td>$-2i\sqrt{\alpha}\gamma y_1$</td>
<td>0</td>
<td>0</td>
<td>2i$\sqrt{\alpha}\beta \gamma h$</td>
<td>0</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$i\sqrt{\alpha}y_1$</td>
<td>$-2i\sqrt{\alpha}\beta h$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2i\sqrt{\alpha}x_3$</td>
<td>$2i\sqrt{\alpha}\beta x_2$</td>
</tr>
<tr>
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<td>0</td>
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<td>$2i\sqrt{\alpha}\gamma x_1$</td>
<td>0</td>
</tr>
</tbody>
</table>
OUR AIM:

Recover the initial algebra of the simple (non-Lie) Malcev algebra 7-dimensional $M(\alpha, \beta, \gamma)$.

**Enveloping algebra $U(M(\alpha, \beta, \gamma))$:**
- alternative algebra generated by $h, x_1, x_2, x_3, y_1, y_2, y_3$
- multiplication $xy$ satisfying relation: $xy - yx = [x, y]$, where commutator $[,]$ is multiplication in $M(\alpha, \beta, \gamma)$
Relations holding in any alternative algebra:

\[(x, y, z) = \frac{1}{6} J(x, y, z), \quad [x, y] \circ (x, y, z) = 0, \quad \forall x, y, z \in U(M(\alpha, \beta, \gamma))\]

Jacobian \(J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]\), associator \((x, y, z) = (xy)z - x(yz)\), Jordan product \(x \circ y = xy + yx\).


**Multiplication table in** \(U(M(\alpha, \beta, \gamma))\):

<table>
<thead>
<tr>
<th></th>
<th>(h)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(y_1)</th>
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</tr>
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<td>(h)</td>
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<td>(-i\sqrt{\alpha}y_1)</td>
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<td>(-i\sqrt{\alpha}y_3)</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(-i\sqrt{\alpha}x_1)</td>
<td>0</td>
<td>(i\sqrt{\alpha}y_3)</td>
<td>(-i\sqrt{\alpha\beta}y_2)</td>
<td>(i\sqrt{\alpha\beta}h)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(-i\sqrt{\alpha}x_2)</td>
<td>(-i\sqrt{\alpha}y_3)</td>
<td>0</td>
<td>(i\sqrt{\alpha\gamma}y_1)</td>
<td>0</td>
<td>(i\sqrt{\alpha\gamma}h)</td>
<td>0</td>
</tr>
<tr>
<td>(x_3)</td>
<td>(-i\sqrt{\alpha}x_3)</td>
<td>(i\sqrt{\alpha\beta}y_2)</td>
<td>(-i\sqrt{\alpha\gamma}y_1)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(i\sqrt{\alpha\beta\gamma}h)</td>
</tr>
<tr>
<td>(y_1)</td>
<td>(i\sqrt{\alpha}y_1)</td>
<td>(-i\sqrt{\alpha\beta}h)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-i\sqrt{\alpha\beta}x_3)</td>
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</tr>
<tr>
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<td>(-i\sqrt{\alpha\beta}x_2)</td>
<td>(i\sqrt{\alpha\gamma}x_1)</td>
<td>0</td>
</tr>
</tbody>
</table>
PROPOSITION

The algebra $U(M(\alpha, \beta, \gamma))$ is alternative $\iff$ it is trivial.
PROPOSITION

The algebra \( U(M(\alpha, \beta, \gamma)) \) is alternative \iff it is trivial.

CONCLUSION:

\[
\begin{align*}
A \text{ associative} & \implies A^- \text{ Lie algebra} \\
A \text{ alternative} & \implies A^- \text{ Malcev algebra}
\end{align*}
\]

Given an simple Malcev algebra \( M \) of finite dimension:

- If \( M \) is Lie algebra then there exists an associative algebra \( A \) such that \( M \subset A^- \)

- If \( M \) is Malcev algebra then does not exist an alternative algebra \( A \) (apart from the trivial) such that \( M \subset A^- \)