The character and the pseudo-character in endomorphism rings

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A short Introduction

Recall that *the character of a point* x from the topological space $(X, \mathcal{T})$ is defined as the smallest number of the form $|\mathcal{B}(x)|$, where $\mathcal{B}(x)$ is a base for $(X, \mathcal{T})$ in x. This cardinal number is denoted by $\chi(x, (X, \mathcal{T}))$.

Now *the character of a topological space* $(X, \mathcal{T})$ is defined as the supremum of the numbers $\chi(x, (X, \mathcal{T}))$ for every $x \in X$. This cardinal is denoted with $\chi((X, \mathcal{T}))$.

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We will call *the pseudo-character of a point* x from a T_1-space X the smallest cardinal number having the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\cap\mathcal{U} = \{x\}$; this cardinal is denoted by $\psi(x, X)$.

The pseudo-character of a T_1-space is defined as the supremum of all numbers $\psi(x, X)$ for every $x \in X$; this cardinal is denoted by $\psi(X)$.

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We will study the following questions:
1) How can we express the character of $\mathrm{End}(A)$ through the invariants of the group A?
2) How can we express the pseudo-character of $\mathrm{End}(A)$ through the invariants of the group A?
3) What are the conditions under which the character of the topological ring $(\mathrm{End}(A), \mathcal{T})$ is the same with the pseudo-character?

The Results

Lemma

Let A be an Abelian group, $A = \oplus_{i\in I} A_i, A_i \neq 0$ *and* $\mathfrak{m}$ *a cardinal number. If* $\chi(\mathrm{End}(A)) \leq \mathfrak{m}$ *then* $|I| \leq \mathfrak{m}$.

Corollary

If A is a direct sum of cyclic groups, then $\chi(\mathrm{End}(A)) \leq \mathfrak{m}$ *if and only if* $|A| \leq \mathfrak{m}$.

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Proposition

If A is a nonzero group of cardinality $\leq \omega$ *and* $\mathfrak{m}$ *an infinite cardinal number, then* $\psi(End(\oplus_{\mathfrak{m}} A)) = \chi(End(\oplus_{\mathfrak{m}} A)) = \mathfrak{m}$.

Corollary

If A is a torsion group, $\mathfrak{m}$ *a cardinal number and* $\chi(\mathrm{End}(A)) \leq \mathfrak{m}$, *then* $|A| \leq 2^{\mathfrak{m}}$.

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A group A is called a *group of Kulikov's type*, if A is the torsion part of the direct product of B_n $(n = 1, 2, \cdots)$. B_n is a direct sum of cyclic groups having the same order p^n.

Theorem

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Lemma

Let A be a reduced p-group and $\mathfrak{m}$ an infinite cardinal number. Then the pseudo-character of $\mathrm{End}(A)$ is $\leq \mathfrak{m}$ if and only if the cardinality of every basic subgroup of A is $\leq \mathfrak{m}$.

Corollary

Let A be a torsion group. Then the pseudo-character of $\mathrm{End}(A)$ is equal to $\aleph_0$ if and only if $A = C \oplus D$ where D is a divisible subgroup with $|D| < \aleph_0$ and $C = \oplus C(p)$, every basic subgroup of $C(p)$ being countable.

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Theorem

Let $A = \oplus_{p \in \mathbb{P}} A_p$ an infinite torsion group, where A_p is the p-primary subgroup of A. Fix for every $p \in \mathbb{P}$ the direct decomposition $A_p = S_p \oplus D_p$, where D_p is the maximal divisible subgroup and S_p a reduced subgroup. Then $\psi(\mathrm{End}(A)) \leq \mathfrak{m}$, where $\mathfrak{m}$ is a cardinal number, if and only if $|D_p| \leq \mathfrak{m}$ and for every basic subgroup B_p of S_p we have $|B_p| \leq \mathfrak{m}$.

Remark

If $A = \oplus_{i \in I} A_i$ where $0 < |A_i| \leq \omega$ and I is infinite, then $\psi(\mathrm{End}(A)) = |I|$.