Maltsev Conditions for Omitting Types

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Six Interesting Families

Quote from "The Structure of Finite Algebra", by Hobby & McKenzie:

"Our theory reveals a sharp division of locally finite varieties of algebras into six interesting new families, each of which is characterized by the behaviour of congruences in the algebras."

Goals of this talk:

- Describe these six families,
- Present various old and new characterizations of them,
- Show that some characterizations are simpler than expected and that
- some of them can not be significantly simplified.

Some Tame Congruence Theory

Tame Congruence Theory

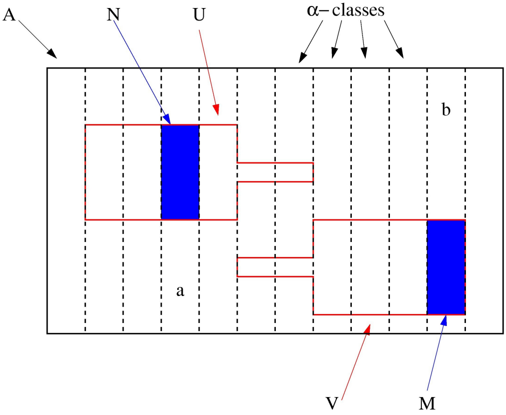
Hobby and McKenzie have developed a notion of neighbourhood, or minimal set of a finite algebra. They show that the behaviour of minimal sets is limited to one of the following five types:

1. Unary
2. Affine
3. 2-element Boolean algebra
4. 2-element Lattice
5. 2-element Semi-lattice

Definition

- We say that a finite algebra $\mathbb{A}$ omits a particular type if no neighbourhoods of that type occur in $\mathbb{A}$.
- A variety $\mathcal{V}$ omits a particular type if each finite member of it does.

Neighbourhoods



Legend

- A is partitioned by the α-classes.
- U and V are α-minimal sets.
- $N = U \cap (a/\alpha)$ and $M = V \cap (b/\alpha)$ are α-neighbourhoods.

Definition

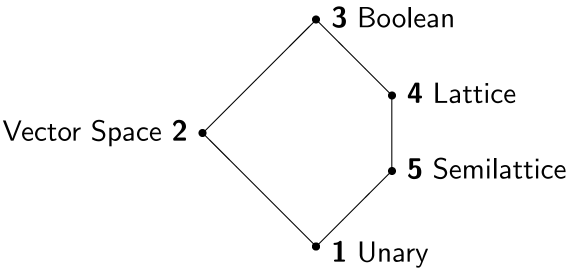
The type of α is equal to the type of any one of the α-neighbourhoods.

An Ordering of Types

Remark

There is a natural order on the five types, determined by the "richness" of the associated algebraic structure:

$\mathbf{1} < \mathbf{2} < \mathbf{3} > \mathbf{4} > \mathbf{5} > \mathbf{1}$



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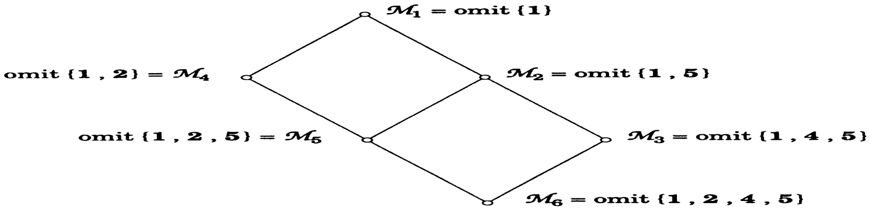
Omitting Types

Remark

With respect to the type ordering, there are six proper order ideals, and for each, Hobby and McKenzie define an associated family of locally finite varieties:



Some Properties of the Six Families

Name	Type Omitting Condition	Other Defining Properties
$\mathcal{M}_1$	$\{\mathbf{1}\}$	largest non-trivial idempotent Maltsev class
$\mathcal{M}_2$	$\{\mathbf{1}, \mathbf{5}\}$	equivalent to satisfying a non-trivial congruence identity
$\mathcal{M}_3$	$\{\mathbf{1}, \mathbf{4}, \mathbf{5}\}$	n-permutable varieties
$\mathcal{M}_4$	$\{\mathbf{1}, \mathbf{2}\}$	congruence meet semi-distributive varieties
$\mathcal{M}_5$	$\{\mathbf{1}, \mathbf{2}, \mathbf{5}\}$	congruence join semi-distributive varieties
$\mathcal{M}_6$	$\{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$	n-permutable and congruence join semi-distributive varieties

Maltsev Conditions

Remark

Each of the six families can be defined in terms of idempotent Maltsev Conditions.

Example

A locally finite variety $\mathcal{V}$ belongs to the class $\mathcal{M}_3$ if and only if for some $n > 0$ there are $\mathcal{V}$-terms $p_i(x, y, z)$, for $1 \leq i \leq n$ such that

$$\begin{aligned} x &\approx p_1(x, y, y), \\ p_i(x, x, y) &\approx p_{i+1}(x, y, y) \quad \text{for each } i, \\ p_n(x, x, y) &\approx y \end{aligned}$$

A locally finite variety $\mathcal{V}$ belongs to the class $\mathcal{M}_4$ if and only if for some $n > 0$ there are $\mathcal{V}$-terms $d_i(x, y, z)$ and $e_i(x, y, z)$, for $1 \leq i \leq n$ such that

$$x \approx d_1(x, x, y),$$

Some Special Terms

Definition

A term $t(x_1, \ldots, x_n)$ of a variety $\mathcal{V}$ is:

- idempotent if the equation $t(x, x, \ldots, x) \approx x$ holds in $\mathcal{V}$,
- a Taylor term if it is idempotent and for each $1 \leq i \leq n$, an equation in the variables $\{x, y\}$ of the form $t(a_1, \ldots, a_n) \approx t(b_1, \ldots, b_n)$ holds in $\mathcal{V}$, where $a_i = x$ and $b_i = y$.
- a weak near unanimity term if it is idempotent and the equations

  $$t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, x, \ldots, x, y)$$

  hold in $\mathcal{V}$,
- a cyclic term if it is idempotent and the equation $t(x_1, x_2, \ldots, x_{n-1}, x_n) \approx t(x_2, x_3, \ldots, x_n, x_1)$ holds in $\mathcal{V}$.

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Omitting the Unary Type

Theorem (Hobby, Maroti, McKenzie)

Let $\mathcal{V}$ be a locally finite variety. The following are equivalent:

- $\mathcal{V} \in \mathcal{M}_1$
- $\mathcal{V}$ *omits the unary type*
- $\mathcal{V}$ *has a Taylor term*
- $\mathcal{V}$ *has a weak near unanimity term*

Theorem (Barto, Kozik)

Let $\mathbb{A}$ be a finite algebra and let $\mathcal{V} = \mathbf{HSP}(\mathbb{A})$. Then $\mathcal{V}$ omits the unary type if and only if for all prime numbers $p > |A|$, $\mathbb{A}$ has a cyclic term of arity p.

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Siggers' Result

Remarks

- *For all $n > 0$ one can find a finite algebra $\mathbb{A}_n$ that has a weak near unanimity term of arity n but of no smaller arity.*
- *From this, it appears that the Maltsev condition for locally finite varieties that omit the unary type is not strong.*
- *but ...*

Theorem (Siggers)

Let $\mathcal{V}$ be a locally finite variety. Then $\mathcal{V}$ omits the unary type if and only if it has a 6-ary idempotent term t such that $\mathcal{V}$ satisfies the equations

$$t(x,x,x,x,y,y) \approx t(x,y,x,y,x,x)$$
$$t(y,y,x,x,x,x) \approx t(x,x,y,x,y,x).$$

Siggers' Result

Remark

Shortly after Siggers announced his result, Markovic and McKenzie observed that 4-ary versions of Siggers' term exist. Here is one version:

Theorem

A locally finite variety $\mathcal{V}$ omits the unary type if and only if it has a 4-ary idempotent term operation t that satisfies the identities:

$$t(y, y, x, x) \approx t(x, y, y, x) \approx t(x, x, x, y).$$

Corollary

The class $\mathcal{M}_1$ is defined by a strong Maltsev condition.

A short proof

Theorem

Let $\mathbb{A}$ be a finite algebra such that $\mathcal{V} = \mathbf{HSP}(\mathbb{A})$ omits the unary type. Then $\mathbb{A}$ has an idempotent term t such that $t(y,y,x,x) \approx t(x,y,y,x) \approx t(x,x,x,y)$.

Proof.

- Let p be some prime number $> |A|$ of the form $5k+3$ for some k,
- let $c(x_1, \ldots, x_p)$ be a cyclic term of $\mathbb{A}$ of arity p,
- Let $t(x,y,z,w) = c(x,x,\ldots,x,y,y,\ldots,y,z,z,\ldots z,w,w,\ldots w)$, where the variables
    - x and z occur $k+1$ times,
    - y occurs k times and
    - w occurs $2k+1$ times.
- c cyclic implies that t satisfies the stated equations.

□

Omitting the Unary and Affine Types

Remarks

- *Recall that the class $\mathcal{M}_4$ consists of all locally finite varieties that omit the unary and affine types.*
- *It was noted earlier that this class is definable by a complicated Maltsev condition.*

Theorem

A locally finite variety omits the unary and affine types if and only if it has 3-ary and 4-ary weak near unanimity terms $v(x, y, z)$ and $w(x, y, z, w)$ that satisfy the equation $v(y, x, x) \approx w(y, x, x, x)$.

Proof.

Uses results of McKenzie and Barto and Kozik on weak near unanimity terms, a result of Barto and Kozik on the constraint satisfaction problem, and a construction of weak near unanimity terms due to Kiss. □

What about the other four families?

Theorem

Of the classes of locally finite varieties $\mathcal{M}_i$, $1 \leq i \leq 6$, only $\mathcal{M}_1$ and $\mathcal{M}_4$ can be defined by strong Maltsev conditions.

Sketch of Proof

- For each n, we construct a finite idempotent algebra $\mathbb{A}_n$ such that $\mathcal{V}_n = \mathbf{HSP}(\mathbb{A}_n)$ omits all types except the Boolean type (type **3**).
- Thus $\mathcal{V}_n$ belongs to all six families.
- Establish that if Σ is any strong Maltsev condition that is satisfied by $\mathcal{V}_n$ for all n, then the variety of semilattices also satisfies Σ.
- Therefore none of the families that omit the semilattice type (type **5**) can be defined by a strong Maltsev condition.

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The algebras $\mathbb{A}_n$

Definition

Let $n > 0$ and $1 \leq i \leq n$.

- Let $\mathbb{A}[i, n]$ be the algebra with universe $\{0, 1\}$ and whose only basic operation is the $2n + 1$-ary operation $t_{(i,n)}$ defined by:

$$t_{(i,n)}(x_0, x_1, \ldots, x_{2n-1}, x_{2n}) = x_0 \wedge (x_1 \wedge x_2) \wedge \cdots \wedge (x_{2i-3} \wedge x_{2i-2}) \wedge (\overline{x_{2i-1}} \vee x_{2i}).$$

- Let $\mathbb{A}_n$ be the cartesian product $\prod_{i=1}^{n} \mathbf{A}[i, n]$ and let $\mathcal{V}_n$ be the variety generated by $\mathbb{A}_n$.[a]

[a] This construction is based on one found in a recent paper by Carvalho, Dalmau, and Krokhin.

Some Observations

Can we do better?

- We've seen that the class $\mathcal{M}_1$ can be characterized by the existence of a 4-ary term. Is it possible that it could also be characterized by the existence of some kind of 3-ary term?
  - No, but
  - it can be characterized by the existence of two 3-ary idempotent terms $p(x, y, z)$ and $q(x, y, z)$ such that

$$p(x, x, y) \approx p(y, x, x) \approx q(x, y, y) \text{ and } p(x, y, x) \approx q(x, y, x).$$

- Something similar happens with $\mathcal{M}_4$, namely, it can be characterized by the existence of three 3-ary idempotent terms that satisfy certain equations. This was observed by M. Maroti and A. Janko.

Conclusion

Conclusions

- Finding "nice" Maltsev conditions for $\mathcal{M}_1$ and $\mathcal{M}_4$ has led to computationally more efficient algorithms to determine if a given finite algebra generates a variety that belongs to one of these classes.
- The study of these Maltsev classes has advanced work on the constraint satisfaction problem (and vice versa).
- In their new book, "The Shape of Congruence Lattices", Kearnes and Kiss study in detail the extensions of the six families to the general case, i.e., the non-locally finite case.
- Question: Can the other four families be defined by better Maltsev conditions than the standard ones?
- Question: Are any other familiar Maltsev conditions equivalent to strong Maltsev conditions for locally finite varieties?