Compressible Modules and Compressible Dimension

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Definition (Zelmanowitz, 1976)

- A nontrivial module M_R is compressible if it can be embedded in any of its nonzero submodules.
- A compressible module is critically compressible if it can not be embedded in any proper factor module.
- A ring is weakly primitive if it possesses a faithful critically compressible module.

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Theorem (Zelmanowitz)

The next statements are equivalent for a ring R.

1. *R is weakly primitive.*
2. *There exists an R-lattice (Δ, V, M) such that, given any elements $m_1, \ldots, m_k \in V$ linerarly independent over Δ, there exists $0 \neq a \in \Delta$ such that, for any elements $n_1, \ldots, n_t \in M$, it can be found $r \in R$ with $an_i = m_i r \in M$ for each $i \in \{1, \ldots, t\}$.*
3. *There exists an R-lattice (Δ, V, M) such that given any $\tau \in \mathrm{End}_\Delta(V)$ and any elements $m_1, \ldots, m_k \in M$ linerarly independent over Δ, there exist $r, s \in R$ with $m_i \tau r = m_i s$ and $0 \neq m_i r \in \Delta m_i$ for each i.*

A natural question

What about the rings?

Can we give a similar definition?

In the way of Zelmanowitz we define

Definition

A ring R is left compressible if it can be embedded in any of its nonzero left ideals.

In that sense we conclude, for instance, that

Theorem

A commutative ring R is compressible if and only if $R[x]$ is compressible.

Another natural question

What can be said about the compressible modules?

- Let ${}_RM$ be a module, then

$$\mathrm{cd}M = \left\{N \leq M \middle| \exists f : M \rightarrowtail N\right\}$$

is called the compressible domain of M, then

- ${}_RM$ is a compressible module if and only if $\mathrm{cd}M = \mathrm{sub}M$, where $\mathrm{sub}M = \{N \leq M\}$;
- ${}_RM$ is incompressible if $\mathrm{cd}M = \{M\}$;
- if $N \leq M$ and $N \in \mathrm{cd}M$, then $\mathrm{cd}N \leq \mathrm{cd}M$ (*i.e.*, cd is order preserving);
- if $N \leq M$, with M quasi-projective then $\mathrm{cd}(M/N) = \left\{Q/N \leq M/N \middle| Q \in \mathrm{cd}M, N \leq Q\right\}$.

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- Consider $T = \{M | M \text{ is incompressible}\}$. Then T is not closed under direct sums.

Example

Let S be a simple module, let us consider $S^{(\mathbb{N})}$ and $S^{(2\mathbb{N})}$ (which are semisimple), then $S^{(\mathbb{N})} \hookrightarrow S^{(2\mathbb{N})}$ in the natural way, so direct sum of simples is not compressible.

Example

If S is a simple module, then S is incompressible.
If V is a finite dimensional vector space, then V is incompressible.

And the ever present short exact sequences ...

Definition

A module ${}_RM$ is absolutely compressible if for any short exact sequence $M \longrightarrow M'' \longrightarrow 0$, then M'' is compressible.

Example

If S is a simple module, then S is absolutely compressible.

Definition

Complementarily, a module $_RM$ is quasi-compressible if for every $N \leq M$, $N \neq 0$, there exists $f : M \to N$, $f \neq 0$

Proposition

If M is compressible and $M \to M'' \to 0$ is a short exact sequence, then M'' is quasi-compressible.

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If M is compressible and $M \to M'' \to 0$ is a short exact sequence, then M'' is quasi-compressible.

A final related definition.

Definition

A module ${}_RM$ is fully invariant compressible (f.i.c.) if for every $N \leq_{f.i.} M$, $N \neq 0$, there exists a monomorphism $f : M \to N$ with $\mathrm{Im} f \leq_{f.i.} N$.

Proposition

- *If $K \leq_{f.i.} M$ and M is f.i.c., then K is f.i.c.*
- *If $\{M_i\}$ is a family of f.i.c. modules, then $\bigoplus M_i$ is f.i.c.*

Thank You!