Compressible Modules and Compressible Dimension

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Definition (Zelmanowitz, 1976)

A nontrivial module $M_R$ is compressible if it can be embedded in any of its nonzero submodules.

A compressible module is critically compressible if it cannot be embedded in any proper factor module.

A ring is weakly primitive if it possesses a faithful critically compressible module.
Definition (Zelmanowitz, 1976)

- A nontrivial module $M_R$ is **compressible** if it can be embedded in any of its nonzero submodules.
- A compressible module is **critically compressible** if it can not be embedded in any proper factor module.
- A ring is **weakly primitive** if it possesses a faithful critically compressible module.
Definition (Zelmanowitz, 1976)

- A nontrivial module $M_R$ is **compressible** if it can be embedded in any of its nonzero submodules.
- A compressible module is **critically compressible** if it cannot be embedded in any proper factor module.
- A ring is **weakly primitive** if it possesses a faithful critically compressible module.
Theorem (Zelmanowitz)

The next statements are equivalent for a ring $R$.

1. $R$ is weakly primitive.

2. There exists an $R$-lattice $(\Delta, V, M)$ such that, given any elements $m_1, \ldots, m_k \in V$ linearly independent over $\Delta$, there exists $0 \neq a \in \Delta$ such that, for any elements $n_1, \ldots, n_t \in M$, it can be found $r \in R$ with $an_i = m_i r \in M$ for each $i \in \{1, \ldots, t\}$.

3. There exists an $R$-lattice $(\Delta, V, M)$ such that given any $\tau \in \text{End}_\Delta(V)$ and any elements $m_1, \ldots, m_k \in M$ linearly independent over $\Delta$, there exist $r, s \in R$ with $m_i \tau r = m_i s$ and $0 \neq m_i r \in \Delta m_i$ for each $i$. 
A natural question

What about the rings?

Can we give a similar definition?
In the way of Zelmanowitz we define

**Definition**

A ring $R$ is *left compressible* if it can be embedded in any of its nonzero left ideals.

In that sense we conclude, for instance, that

**Theorem**

A commutative ring $R$ is compressible if and only if $R[x]$ is compressible.
What can be said about the compressible modules?
Let \( R^\times M \) be a module, then

\[
\text{cd} M = \{ N \leq M \mid \exists f : M \rightarrow N \}
\]

is called the compressible domain of \( M \), then

- \( R^\times M \) is a compressible module if and only if \( \text{cd} M = \text{sub} M \), where \( \text{sub} M = \{ N \leq M \} \);
- \( R^\times M \) is incompressible if \( \text{cd} M = \{ M \} \);
- if \( N \leq M \) and \( N \in \text{cd} M \), then \( \text{cd} N \leq \text{cd} M \) (i.e., \( \text{cd} \) is order preserving);
- if \( N \leq M \), with \( M \) quasi-projective then
  \[
  \text{cd}(M/N) = \{ Q/N \leq M/N \mid Q \in \text{cd} M, N \leq Q \}.
  \]
Let $R^M$ be a module, then

$$\text{cd}M = \{ N \leq M \, | \exists f : M \rightarrow N \}$$

is called the compressible domain of $M$, then $R^M$ is a compressible module if and only if $\text{cd}M = \text{sub}M$, where $\text{sub}M = \{ N \leq M \}$;

$R^M$ is incompressible if $\text{cd}M = \{ M \}$;

if $N \leq M$ and $N \in \text{cd}M$, then $\text{cd}N \leq \text{cd}M$ (i.e., cd is order preserving);

if $N \leq M$, with $M$ quasi-projective then

$$\text{cd}(M/N) = \{ Q/N \leq M/N \, | \, Q \in \text{cd}M, N \leq Q \}.$$
Let $\mathcal{R}M$ be a module, then
\[
\text{cd}M = \{ N \leq M \mid \exists f : M \to N \}
\]
is called the compressible domain of $M$, then
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\mathcal{R}M \text{ is a compressible module if and only if } \text{cd}M = \text{sub}M,
\]
where $\text{sub}M = \{ N \leq M \}$;
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if $N \leq M$ and $N \in \text{cd}M$, then $\text{cd}N \leq \text{cd}M$ (i.e., cd is order preserving);
if $N \leq M$, with $M$ quasi-projective then
\[
\text{cd}(M/N) = \{ Q/N \leq M/N \mid Q \in \text{cd}M, N \leq Q \}. \]
Let $R M$ be a module, then

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\text{cd} M = \{ N \leq M \mid \exists f : M \rightarrow N \}
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is called the compressible domain of $M$, then

- $R M$ is a compressible module if and only if $\text{cd} M = \text{sub} M$, where $\text{sub} M = \{ N \leq M \}$;
- $R M$ is incompressible if $\text{cd} M = \{ M \}$;
- if $N \leq M$ and $N \in \text{cd} M$, then $\text{cd} N \leq \text{cd} M$ (i.e., $\text{cd}$ is order preserving);
- if $N \leq M$, with $M$ quasi-projective then
  $$
  \text{cd} (M/N) = \{ Q/N \leq M/N \mid Q \in \text{cd} M, N \leq Q \}.
  $$
Let $\mathcal{R}M$ be a module, then

$$\text{cd}M = \{ N \leq M | \exists f : M \rightarrow N \}$$

is called the compressible domain of $M$, then

- $\mathcal{R}M$ is a compressible module if and only if $\text{cd}M = \text{sub}M$, where $\text{sub}M = \{ N \leq M \}$;
- $\mathcal{R}M$ is incompressible if $\text{cd}M = \{ M \}$;
- if $N \leq M$ and $N \in \text{cd}M$, then $\text{cd}N \leq \text{cd}M$ (i.e., $\text{cd}$ is order preserving);
- if $N \leq M$, with $M$ quasi-projective then
  $$\text{cd}(M/N) = \{ Q/N \leq M/N | Q \in \text{cd}M, N \leq Q \}.$$
Consider $T = \{ M \mid M \text{ is incompressible} \}$. Then $T$ is not closed under direct sums.

**Example**

Let $S$ be a simple module, let us consider $S^{(N)}$ and $S^{(2N)}$ (which are semisimple), then $S^{(N)} \hookrightarrow S^{(2N)}$ in the natural way, so direct sum of simples is not compressible.

**Example**

If $S$ is a simple module, then $S$ is incompressible. If $V$ is a finite dimensional vector space, then $V$ is incompressible.
And the ever present short exact sequences ...
Definition

A module \(R M\) is **absolutely compressible** if for any short exact sequence \(M \rightarrow M'' \rightarrow 0\), then \(M''\) is compressible.

Example

If \(S\) is a simple module, then \(S\) is absolutely compressible.
Definition

Complementarily, a module $R^M$ is quasi-compressible if for every $N \leq M$, $N \neq 0$, there exists $f : M \to N$, $f \neq 0$

Proposition

If $M$ is compressible and $M \to M'' \to 0$ is a short exact sequence, then $M''$ is quasi-compressible.
Definition
Complementarily, a module $\mathcal{R}M$ is quasi-compressible if for every $N \leq M, N \neq 0$, there exists $f : M \to N, f \neq 0$

Proposition
If $M$ is compressible and $M \to M'' \to 0$ is a short exact sequence, then $M''$ is quasi-compressible.
A final related definition.
Definition
A module $\mathbb{R}M$ is fully invariant compressible (f.i.c.) if for every $N \leq_{f.i.} M$, $N \neq 0$, there exists a monomorphism $f : M \to N$ with $\text{Im} f \leq_{f.i.} N$.

Proposition
- If $K \leq_{f.i.} M$ and $M$ is f.i.c., then $K$ is f.i.c.
- If $\{M_i\}$ is a family of f.i.c. modules, then $\bigoplus M_i$ is f.i.c.
Thank You!