Semicoprime Preradicals

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Contents

Notation

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- R is an associative ring with unit.
- $R\text{-Mod}$ is the category of left R-modules.
- $R\text{-simp}$ is a complete irredundant set of representatives of the isomorphism classes of simple left R-modules.

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1. $\sigma M \leq M$ for each $M \in R\text{-Mod}$.
2. For each homomorphism $f : M \to N$, $f(\sigma M) \leq \sigma N$.

R-pr as a complete (big) lattice

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Note: Join and meet can be defined
for arbitrary classes of preradicals.

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$$\sigma^2 = \sigma\sigma$$
$$\sigma_2 = (\sigma : \sigma)$$

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σ is *idempotent* if $\sigma^2 = \sigma$.
σ is a *radical* if $\sigma_2 = \sigma$.
σ is *nilpotent* if $\sigma^n = 0$ for some n.
σ is *unipotent* if $\sigma_n = 1$ for some n.

alpha and omega preradicals

Definition

Let $M \in R\text{-}\mathrm{Mod}$. A submodule N of M is called *fully invariant* (written $N \leq_{fi} M$) if for each endomorphism $f : M \to M$ we have $f(N) \leq N$

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Let $K \in R\text{-}\mathrm{Mod}$.

$$\alpha_N^M(K) = \sum\{f(N) \mid f \in Hom_R(M, K)\}$$

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Let $K \in R\text{-Mod}$.

$$\alpha_N^M(K) = \sum\{f(N) \mid f \in Hom_R(M, K)\}$$

$$\omega_N^M(K) = \bigcap\{f^{-1}(N) \mid f \in Hom_R(K, M)\}$$

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some properties

Proposition

If $\sigma \in R$-pr then:
$\sigma = \bigvee\{\alpha^{M}_{\sigma M} \mid M \in R\text{-Mod}\} = \bigwedge\{\omega^{M}_{\sigma M} \mid M \in R\text{-Mod}\}$.

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Proposition

If $\sigma \in R\text{-pr}$ and $M, N \in R\text{-Mod}$ then:
$\sigma(M) = N \iff N \leq_{fi} M$ and $\alpha^M_N \preceq \sigma \preceq \omega^M_N$.

atoms and coatoms in $R\text{-}\mathrm{pr}$

Theorem

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The set of atoms is $\{\alpha_S^{ES} | S \in R\text{-simp}\}$.

The set of coatoms is $\{\omega_I^R | I \text{ is a maximal ideal of } R\}$.

some definitions on primeness for preradicals

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1. *prime* if $\sigma \neq 1$ and for any $\tau, \eta \in R\text{-pr}$ $\tau\eta \preceq \sigma$ implies $\tau \preceq \sigma$ or $\eta \preceq \sigma$.

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4. *semicoprime* if $\sigma \neq 0$ and for any $\tau \in R\text{-pr}$ $\sigma \preceq \tau_2$ implies $\sigma \preceq \tau$.

semicoprime preradicals: basic properties

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Let $\sigma \in R\text{-pr}$ and $\{\sigma_i\}_{i \in I} \subseteq R\text{-pr}$. Then:

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Example

For each maximal ideal I of R, $\alpha_{R/I}^{R/I}$ is a coprime preradical. Therefore $\bigvee\{\alpha_{R/I}^{R/I} \mid I \text{ maximal ideal of } R\}$ is semicoprime.

product and coproduct of submodules

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Notation: $K^2 = KK$
$K_2 = (K : K)$.

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two theorems involving modules and preradicals

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Let $M \in R\text{-Mod}$ and let $0 \neq N \leq_{fi} M$. The following conditions are equivalent:

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Theorem

Let $M \in R\text{-Mod}$ be such that for each $N \leq_{fi} M$ we have $(\omega_N^M)_2 = \omega_{(N:N)}^M$. If $\sigma \in R\text{-pr}$ is semicoprime and $\sigma(M) \neq 0$ then $\sigma(M)$ is semicoprime in M.

comparing three preradicals

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Then $\eta \preceq \sigma^0 \preceq \nu_0$.

a characterization of rings

Theorem

For a ring R the following conditions are equivalent:

1. R is a finite product of simple rings.

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4. ${}_R R$ is a semicoprime module.

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2. Each coatom of R-pr is a radical.
3. 1 is a semicoprime preradical.
4. ${}_RR$ is a semicoprime module.
5. $\sigma^0 = 1$.

two operators on R-pr

Definition

Let $\tau \in R\text{-pr}$. We define:

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2. For each radical ρ we have $\overline{C(\rho)} \preceq \rho$.

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2. For each radical ρ we have $\overline{C(\rho)} \preceq \rho$.
3. For each semicoprime preradical σ we have $\sigma \preceq C(\overline{\rho})$.

a Galois connection between $R\text{-}\mathrm{scp}$ and $R\text{-}\mathrm{rad}$

Notation:

$R\text{-}\mathrm{scp}$ denotes the class of all semicoprime preradicals.
$R\text{-}\mathrm{rad}$ denotes the class of all radicals.

a Galois connection between R-scp and R-rad

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$R\text{-scp}$ denotes the class of all semicoprime preradicals.
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Theorem

$\overline{(\ \ )} : R\text{-scp} \to R\text{-rad}$ and $C : R\text{-rad} \to R\text{-scp}$ form a Galois connection between those ordered classes of preradicals.

a Galois connection between R-scp and R-rad

Notation:

R-scp denotes the class of all semicoprime preradicals.
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Theorem

$\overline{(\ \ )} : R\text{-scp} \to R\text{-rad}$ and $C : R\text{-rad} \to R\text{-scp}$ form a Galois connection between those ordered classes of preradicals.

Corollary

1. $C\overline{(\ \ )}$ is a closure operator on R-scp .

a Galois connection between $R\text{-scp}$ and $R\text{-rad}$

Notation:

$R\text{-scp}$ denotes the class of all semicoprime preradicals.
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Theorem

$\overline{(\ \ )} : R\text{-scp} \to R\text{-rad}$ and $C : R\text{-rad} \to R\text{-scp}$ form a Galois connection between those ordered classes of preradicals.

Corollary

1. $C\overline{(\ \ )}$ is a closure operator on $R\text{-scp}$.
2. $\overline{C(\ \ )}$ is an interior operator on $R\text{-rad}$.

some references

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