On congruence lattices of nilsemigroups

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Dedicated to Jaroslav Ježek

Definitions

Let L be a complete lattice and $a \in L$.

- a is *compact* if, for any $X \subseteq L$ with $a \leqslant \bigvee X$, there exists a finite subset $X' \subseteq X$ such that $a \leqslant \bigvee X'$.
- L is *algebraic* if every its element is the join of some compact elements.

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- W.A. Lampe (1982) *If the unit of an algenraic lattice L is compact, then L is represented as the congruence lattice of some groupoid.*

Representation of distributive algebraic lattices

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  F. Wehrung (2007) *There exists a distributive algebraic lattice which is not isomorphic to the congruence lattice of any semilattice.*

Representation of distributive algebraic lattices

- - P. Ružička, J. Tůma, F. Wehrung (2005) *Every distributive algebraic lattice with cardinality of the set of compact elements not greater than* $\aleph_1$ *is isomorphic to the lattice of normal subgroups of some group.*

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- A.P., V.Repnitskiĭ (2009) *Every distributive algebraic lattice with the set of compact elements being a sublattice with unit is isomorphic to the congruence lattice of a suitable semigroup.*

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- Theorem 2 (A.P., V.Repnitskiĭ (2010)).
  Every distributive algebraic lattice with cardinality of the set of compact elements not greater than $\aleph_0$ is isomorphic to the congruence lattice of some 2-nilsemigroup.

Distance function

- S is a semigroup. P is a $(\vee, 0)$-semilattice.
  A mapping $\delta : S \times S \to P$ is called a semigroup distance function if
  1) $\delta(x, x) = 0$ for all $x \in S$,
  2) $\delta(x, y) = \delta(y, x)$ for all $x, y \in S$,
  3) $\delta(x, z) \leqslant \delta(x, y) \vee \delta(y, z)$ for all $x, y, z \in S$,
  4) $\delta(xs, yt) \leqslant \delta(x, y) \vee \delta(s, t)$ for all $x, y, s, t \in S$.

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- I is an ideal in P.
  $O_\delta(I) = \{(x, y) \in S \times S : \delta(x, y) \in I\}$.
  $O_\delta(a) = O_\delta(< a >)$.

Proposition 1.
Let $\delta : S \times S \longrightarrow P$ be a semigroup distance function and the following conditions hold:
1) for all $a, b \in P$ and $x, y \in S$, if $\delta(x, y) \leqslant a \vee b$, then $(x, y) \in O_\delta(a) \vee O_\delta(b)$,
2) δ is surjective,
3) for all $(a, b), (c, d) \in S^2$, if $\delta(a, b) \leqslant \delta(c, d)$, then $(a, b) \in O_\delta(0) \vee \Theta(c, d)$.
Then the mapping $O_\delta : J(P) \longrightarrow ConS$ is an isomorphism $J(P)$ onto $[O_\delta(0), S \times S] \cong Con\ S/O_\delta(0)$.

Proposition 2.
Let P be a lattice with unit. Let S be a 2-nilsemigroup and $\delta : S \times S \to P$ a semigroup distance function.
Let $\delta(a, b) \leqslant \delta(c, d)$ for $a, b, c, d \in S$.
Then there exist a 2-nilsemigroup $\tilde{S}$ and a semigroup distance function $\tilde{\delta} : \tilde{S} \times \tilde{S} \to P$ such that
S is a subsemigroup in $\tilde{S}$, $\tilde{\delta}|_{S \times S} = \delta$ and
$(a, b) \in O_{\tilde{\delta}}(0) \vee \Theta(c, d)$ in $\tilde{S}$.

Technique

- $\bar{S} = S * F(u, v, w, \bar{u}, \bar{v}, \bar{w})$
  $I = \{x \in \bar{S} | x = smzmt$ or $x = s0t,$
  where $m \in \{u, v, w, \bar{u}, \bar{v}, \bar{w}\}$ and $s, z, t \in \bar{S}\}$.

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  where $m \in \{u, v, w, \bar{u}, \bar{v}, \bar{w}\}$ and $s, z, t \in \bar{S}\}$.
- $\tilde{S} = \bar{S}/I$ is a 2-nilsemigroup.

$G = G(\tilde{S}, E)$

$p \overset{e}{\longleftrightarrow} q$ if and only if

- 0) $p = q$ and $e = 0$;
- 1) $p = sxt$, $q = syt$ and $e = \delta(x, y)$ for some $s, t \in \tilde{S}^1$ and $x, y \in S$;
- 2) $p = sux\bar{u}t$, $q = sat$ and $e = \delta(x, c)$ for some $s, t \in \tilde{S}^1$ and $x \in S$ (or symmetrically);
- 3) $p = swx\bar{w}t$, $q = sbt$ and $e = \delta(x, d)$ for some $s, t \in \tilde{S}^1$ and $x \in S$ (or symmetrically);
- 4) $p = sux\bar{u}t$, $q = svy\bar{v}t$ or
  $p = svx\bar{v}t$, $q = swy\bar{w}t$,
  and $e = \delta(x, d) \vee \delta(y, c)$ for some $s, t \in \tilde{S}^1$ and
  $(x, y) \in S \times S$ (or symmetrically).

Let P be a path $p = \ p_0 \xleftrightarrow{e_1} p_1 \xleftrightarrow{e_2} p_2 \xleftrightarrow{e_3} \cdots \xleftrightarrow{e_n} p_n = q$.
Define $e(P) = \bigvee e_i$.

$\tilde{\delta}(p, q) = \bigwedge\{e(P) | P \text{ is a path from p to q}\}.$

We have
$\tilde{\delta}(a, uc\bar{u}) = \tilde{\delta}(ud\bar{u}, vc\bar{v}) = \tilde{\delta}(vd\bar{v}, wc\bar{w}) = \tilde{\delta}(wd\bar{w}, b) = 0;$
$(uc\bar{u}, ud\bar{u}), (vc\bar{v}, vd\bar{v}), (wc\bar{w}, wd\bar{w}) \in \Theta(c, d);$
so $(a, b) \in O_{\tilde{\delta}} \vee \Theta(c, d)$.

Ju. Ershov (1977) and P. Pudlák (1985) *Every distributive* $(\vee, 0)$*-semilattice is the directed union of its finite distributive* $(\vee, 0)$*-subsemilattices.*

This is equivalent to the following:

Any finite subset of a distributive $(\vee, 0)$*-semilattice* P *is included into a finite distributive* $(\vee, 0)$*-subsemilattice of* P.

References

1. R. Freese, W. A. Lampe and W. Taylor, *Congruence lattices of algebras of fixed similarity type. I*, Pacific J. Math. **82** (1979), 59–68.

2. G. Grätzer, *General Lattice Theory. Second edition, new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, F. Wehrung, and R. Wille*, Birkhäuser Verlag, Basel, 1998, xix+663 p.p.

3. G. Grätzer and E. T. Schmidt, *Characterizations of congruence lattices of abstract algebras*, Acta Sci. Math. (Szeged), **24** (1963), 34–59.

4. W. A. Lampe, *Congruence lattices of algebras of fixed similarity type, II*, Pacific J. Math. **103** (1982), 475–508.

5. W. A. Lampe, *Results and problems on congruence lattice representations*, Algebra univers. **55** (2006), 127–135.

6. P. Pudlák, *On congruence latticees of lattices*, Algebra Universalis 20(1985), 96-114.
7. V. Repnitskiĭ and J. Tuma, *Intervals in subgroup lattices of countable locally finite groups*, Algebra univers. **59** (2008), 49–71.
8. P. Ružička, J. Tuma and F. Wehrung, *Functorial liftings of distributive semilattices by distances of small type*, Laboratoire de Mathématiques, Nicolas Oresme, Rapport de recherche-2005-22, Université de Caen, CNRS UMR 6139, 20 p.
9. E. T. Schmidt, *The ideal lattice of a distributive lattice with* 0 *is the congruence lattice of a lattice*, Acta Sci. Math. (Szeged) **43** (1981), 153–168.
10. J. Tůma, *Semilattice-valued measures*, Preprint: November 1, Submitted to Contr.Gen.Alg.18 (2007), 12 p.
11. F. Wehrung, *A solution to Dilworth's congruence lattice problem*, Advances in Mathematics **216** (2007), 610-625.