Commuting polynomial functions over distributive lattices

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joint work with
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Commuting operations

Let A be an arbitrary set, and n and m positive integers.

We denote $[n] := \{1, \dots, n\}$.

Definition

We say that $f\colon A^n \to A$ and $g\colon A^m \to A$ commute if

$$f\big(g(a_{11}, a_{12}, \dots, a_{1m}), \dots, g(a_{n1}, a_{n2}, \dots, a_{nm})\big) \\ = g\big(f(a_{11}, a_{21}, \dots, a_{n1}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})\big),$$

for all $a_{ij} \in A$ ($i \in [n]$, $j \in [m]$).

If f and g commute, then we write $f \perp g$.

Commuting operations

In other words, f and g commute if

$$
\begin{array}{cccccccc}
 & f(& f(& & f(& & & f(\\
g(& a_{11} & a_{12} & \cdots & a_{1m} &) & = & c_1 \\
g(& a_{21} & a_{22} & \cdots & a_{2m} &) & = & c_2 \\
 & \vdots & \vdots & \ddots & \vdots & & & \vdots \\
g(& a_{n1} & a_{n2} & \cdots & a_{nm} &) & = & c_n \\
 &) &) & &) & & &) \\
 & \parallel & \parallel & & \parallel & & & \parallel \\
g(& d_1 & d_2 & \cdots & d_m &) & = & b
\end{array}
$$

A particular case . . .

For $n = m = 2$, we have $f \perp g$ if

$$f\big(g(a_{11}, a_{12}), g(a_{21}, a_{22})\big) = g\big(f(a_{11}, a_{21}), f(a_{12}, a_{22})\big).$$

Theorem (Eckmann–Hilton, 1962)

If f and g are binary operations on A with an identity element and $f \perp g$, then $f = g$ and $(A; f)$ is a commutative monoid.

The relevance of commutation in universal algebra

Commutation is the defining property of:

1. entropic algebras,
2. modes,
3. centralizer clones,
4. ...

Self-commuting operations

Let A be an arbitrary set, and n a positive integer.

Definition

An operation $f\colon A^n \to A$ is self-commuting (or bisymmetric) if $f \perp f$, that is,

$$f\big(f(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, f(a_{n1}, a_{n2}, \ldots, a_{nn})\big) \\ = f\big(f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{1n}, a_{2n}, \ldots, a_{nn})\big),$$

for every $a_{ij} \in A$.

A particular case . . .

An algebra $(A; f)$ where f is a binary operation that satisfies the identity

$$f\big(f(a_{11}, a_{12}), f(a_{21}, a_{22})\big) = f\big(f(a_{11}, a_{21}), f(a_{12}, a_{22})\big)$$

is called a medial groupoid.

Thus, the notion of self-commutation generalizes mediality.

Lattice polynomial functions

Let $(L; \wedge, \vee)$ be a lattice with least and greatest elements 0 and 1, respectively.

Definition

A (lattice) polynomial function is any map $p : L^n \to L$ which is a composition of

1. the lattice operations $\wedge$, $\vee$,
2. projections $\mathbf{x} \mapsto x_i$, $i \in [n]$, and
3. constant functions $\mathbf{x} \mapsto c$, $c \in L$.

Representations: disjunctive normal form

A function $p\colon L^n \to L$ has a disjunctive normal form (**DNF**) if

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i\Big)$$

for some $a_I \in L$ $(I \subseteq [n])$.

Representations: disjunctive normal form

Proposition (Goodstein 1965)

Let $(L; \wedge, \vee)$ be a bounded distributive lattice. A function $p\colon L^n \to L$ is a polynomial function if and only if it has the **DNF**

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(p(\mathbf{e}_I) \wedge \bigwedge_{i \in I} x_i \Big),$$

where for $I \subseteq [n]$, $\mathbf{e}_I \in \{0,1\}^n$ is the characteristic vector of I:

$$(\mathbf{e}_I)_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

Corollary

Let L be a bounded distributive lattice. Every polynomial function $p\colon L^n \to L$ is uniquely determined by its restriction to $\{0,1\}^n$.

Corollary

Every polynomial function $p\colon L^n \to L$ over a bounded distributive lattice L has a **DNF**

$$p(\mathbf{x}) = \bigvee_{I \subseteq [n]} \Big(a_I \wedge \bigwedge_{i \in I} x_i\Big),$$

where $a_I \leq a_J$ whenever $I \subseteq J$.

Our problems

Problem

Give necessary and sufficient conditions for two lattice polynomial functions to commute.

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Problem

Explicitly describe the self-commuting lattice polynomial functions.

Commuting lattice polynomial functions

Theorem (Behrisch, Kearnes, Lehtonen, Szendrei 2010)

Let L be a bounded distributive lattice, and let $f\colon L^m \to L$ and $g\colon L^n \to L$ be polynomial functions over L, given by the DNFs

$$f = \bigvee_{S \subseteq [m]} a_S \wedge \bigwedge_{i \in S} x_i, \qquad g = \bigvee_{T \subseteq [n]} b_T \wedge \bigwedge_{i \in T} x_i.$$

The following are equivalent:

(i) $f \perp g$.

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The following are equivalent:

(ii) For all $U_1, U_2 \subseteq [m]$, $V_1, V_2 \subseteq [n]$,

$$a_\emptyset \vee a_{[m]} b_\emptyset \vee a_{U_1 \cap U_2} b_{V_1 \cup V_2} \vee a_{U_1} b_{V_1} \vee a_{U_2} b_{V_2} \vee a_{U_1 \cup U_2} b_{V_1} b_{V_2} =$$
$$b_\emptyset \vee b_{[n]} a_\emptyset \vee b_{V_1 \cap V_2} a_{U_1 \cup U_2} \vee b_{V_1} a_{U_1} \vee b_{V_2} a_{U_2} \vee b_{V_1 \cup V_2} a_{U_1} a_{U_2}.$$

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$$a_\emptyset \vee a_{U_1} a_{U_2} b_{V_1 \cup V_2} = a_\emptyset \vee a_{U_1 \cap U_2} b_{V_1 \cup V_2} \vee a_{U_1} a_{U_2} (b_{V_1} \vee b_{V_2}),$$
$$b_\emptyset \vee b_{V_1} b_{V_2} a_{U_1 \cup U_2} = b_\emptyset \vee b_{V_1 \cap V_2} a_{U_1 \cup U_2} \vee b_{V_1} b_{V_2} (a_{U_1} \vee a_{U_2}).$$

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The following are equivalent:

(iv) $a_\emptyset \vee b_\emptyset \leq a_{[m]} b_{[n]}$ and for all $U_1, \ldots, U_p \subseteq [m]$ ($p \geq 1$), $V_1, \ldots, V_q \subseteq [n]$ ($q \geq 1$),

$$a_\emptyset \vee ((\bigwedge_{i=1}^{p} a_{U_i}) b_{\bigcup_{j=1}^{q} V_j}) = a_\emptyset \vee (a_{\bigcap_{i=1}^{p} U_i} b_{\bigcup_{j=1}^{q} V_j}) \vee ((\bigwedge_{i=1}^{p} a_{U_i})(\bigvee_{j=1}^{q} b_{V_j})),$$

$$b_\emptyset \vee ((\bigwedge_{j=1}^{q} b_{V_j}) a_{\bigcup_{i=1}^{p} U_i}) = b_\emptyset \vee (b_{\bigcap_{j=1}^{q} V_j} a_{\bigcup_{i=1}^{p} U_i}) \vee ((\bigwedge_{j=1}^{q} b_{V_j})(\bigvee_{i=1}^{p} a_{U_i})).$$

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(v) $a_\emptyset \vee b_\emptyset \leq a_{[m]} b_{[n]}$ and for all $U, U_1, \ldots, U_p \subseteq [m]$ $(p \geq 1)$, $V, V_1, \ldots, V_q \subseteq [n]$ $(q \geq 1)$,

$$a_\emptyset \vee \Big(\big(\bigwedge_{i=1}^{p} a_{U_i}\big) b_V\Big) = a_\emptyset \vee \big(a_{\bigcap_{i=1}^{p} U_i} b_V\big) \vee \Big(\big(\bigwedge_{i=1}^{p} a_{U_i}\big)\big(\bigvee_{v \in V} b_v\big)\Big),$$

$$b_\emptyset \vee \Big(\big(\bigwedge_{j=1}^{q} b_{V_j}\big) a_U\Big) = b_\emptyset \vee \big(b_{\bigcap_{j=1}^{q} V_j} a_U\big) \vee \Big(\big(\bigwedge_{j=1}^{q} b_{V_j}\big)\big(\bigvee_{u \in U} a_u\big)\Big).$$

Corollaries

If we take $f = g$, our theorem gives a characterization of self-commuting lattice polynomial functions.

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If we place extra assumptions on the underlying lattice L, we can get more stringent conditions.

Self-commuting lattice polynomial functions

Theorem

Let L be a bounded distributive lattice, and let $f\colon L^m \to L$ be a polynomial function over L, given by the DNF

$$f = \bigvee_{S \subseteq [m]} a_S \wedge \bigwedge_{i \in S} x_i.$$

The following are equivalent:

(i) $f \perp f$.

Self-commuting lattice polynomial functions

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The following are equivalent:

(ii) For all $U_1, U_2, V_1, V_2 \subseteq [m]$,

$$a_{U_1 \cap U_2} a_{V_1 \cup V_2} \vee a_{U_1} a_{V_1} \vee a_{U_2} a_{V_2} \vee a_{U_1 \cup U_2} a_{V_1} a_{V_2} =$$
$$a_{U_1} a_{U_2} a_{V_1 \cup V_2} \vee a_{U_1} a_{V_1} \vee a_{U_2} a_{V_2} \vee a_{U_1 \cup U_2} a_{V_1 \cap V_2}.$$

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The following are equivalent:

(iii) For all $U_1, U_2, V_1, V_2 \subseteq [m]$,

$$a_{U_1} a_{U_2} a_{V_1 \cup V_2} = a_{U_1 \cap U_2} a_{V_1 \cup V_2} \vee a_{U_1} a_{U_2} (a_{V_1} \vee a_{V_2}).$$

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The following are equivalent:

(iv) For all $U_1, \ldots, U_p, V_1, \ldots, V_q \subseteq [m]$ ($p \geq 1$, $q \geq 1$),

$$\Big(\big(\bigwedge_{i=1}^{p} a_{U_i}\big) a_{\bigcup_{j=1}^{q} V_j}\Big) = \Big(a_{\bigcap_{i=1}^{p} U_i} a_{\bigcup_{j=1}^{q} V_j}\Big) \vee \Big(\big(\bigwedge_{i=1}^{p} a_{U_i}\big)\big(\bigvee_{j=1}^{q} a_{V_j}\big)\Big).$$

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The following are equivalent:

(v) For all $U_1, \ldots, U_p, V \subseteq [m]$ $(p \geq 1)$,

$$((\bigwedge_{i=1}^{p} a_{U_i}) a_V) = a_\emptyset \vee (a_{\bigcap_{i=1}^{p} U_i} a_V) \vee ((\bigwedge_{i=1}^{p} a_{U_i})(\bigvee_{v \in V} a_v)).$$

Special case: L is a chain

Theorem (Couceiro, Lehtonen 2010)

Let $(L; \wedge, \vee)$ be a bounded chain. A polynomial function $f\colon L^n \to L$ is self-commuting if and only if

$$f = a_\emptyset \vee \bigvee_{i \in [n]} (a_i \wedge x_i) \vee \bigvee_{1 \leq \ell \leq r} \Big(a_{S_\ell} \wedge \bigwedge_{i \in S_\ell} x_i\Big),$$

where $r \geq 0$, $|S_1| \geq 2$, and

1. $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r \subseteq [n]$, and
2. if $r \geq 1$, then for all $i \in [n]$, there is a $j \in S_1$ such that $a_i \leq a_j$.

Example 1

Consider $f\colon [0,1]^3 \to [0,1]$ given by $\quad f = (x_1 \wedge x_2) \vee (x_2 \wedge x_3)$.

$$
\begin{array}{cccccccc}
 & f(& f(& f(& & & & f(\\
f(& 0 & 1 & 1 &) = & & & 1 \\
f(& 1 & 1 & 0 &) = & & & 1 \\
f(& 0 & 0 & 0 &) = & & & 0 \\
 &) &) &) & & & &) \\
 & \| & \| & \| & & & & \| \\
f(& 0 & 1 & 0 &) = & \color{red}{0} & \color{red}{\neq} & \color{red}{1}
\end{array}
$$

Thus f is not self-commuting!

Example 2

Consider $f\colon [0,1]^3 \to [0,1]$ given by $f = (0.5 \wedge x_1) \vee (x_2 \wedge x_3)$.

$$
\begin{array}{lccccccc}
 & f(& f(& f(& & & & f(\\
f(& 0 & 0 & 1 &) = & & & 0 \\
f(& 0 & 1 & 0 &) = & & & 0 \\
f(& 0 & 1 & 1 &) = & & & 1 \\
 &) &) &) & & & &) \\
 & \parallel & \parallel & \parallel & & & & \parallel \\
f(& 0 & 1 & 0.5 &) = & 0.5 & \neq & 0
\end{array}
$$

Thus f is not self-commuting!

Thank you for your attention!