On non-negative integer quadratic forms

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Outline

Integer quadratic forms

An integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$

$$q(x) = \sum_{i \in \overline{1,n}} q_i x_i^2 + \sum_{i<j} q_{ij} x_i x_j, \quad (q_i, q_{ij} \in \mathbb{Z}, q_{ij} = q_{ji})$$

- is semi-integer if $q_{ij} \in q_i\mathbb{Z}$ for all $i, j \in \overline{1, n}$
- is integer if $\frac{q_{ij}}{q_i} \in \mathbb{Z}$ for all $i \in \overline{1, n}$
- is semi-unit if $q_i \in \{0, 1\}$ for all $i \in \overline{1, n}$
- is unit if $q_i = 1$ for all $i \in \overline{1, n}$
- is classic if $q_i > 0$ and $q_{ij} \leqslant 0$ for all $i, j \in \overline{1, n}$

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Matrix A

Denote by $(\ ,\ ) = (\ ,\ )_q : \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{2}\mathbb{Z}$
the associated symmetrical bilinear form
$(x, y)_q = \frac{1}{2}(q(x+y) - q(x) - q(y))$, $x, y \in \mathbb{Z}^n$.
Let $R = \{\alpha_1, \dots, \alpha_n\}$ be canonical base of $\mathbb{Z}^n$.

$$A_q = \left(A_{ij}\right)_{i,j \in \overline{1,n}}$$

$A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ if $q_i \neq 0$ and $A_{ij} = 0$ otherwise.

Reductions

Canonical base R is called the simple root base of q.

Given a root base R of $\mathbb{Z}^n$, the form $q : \mathbb{Z}^n \to \mathbb{Z}$, and $i, j \in \overline{1, n}$, we construct the new root base $R' = \{\alpha'_k, k \in \overline{1, n}\}$ of $\mathbb{Z}^n$:

$$\alpha'_k = \alpha_k,\ k \neq r,$$
$$\alpha'_r = \alpha_r + \lambda \alpha_s.$$

Then the form q' is uniquely defined.

If $\lambda = -A_{sr}$ correspondent form transformation R^{+}_{sr} is a Gabrielov transformation or reduction.

If $\lambda = -1$ and $A_{sr} > 0$ it is inflation

If $\lambda = 1$ and $A_{rs} < 0$ it is deflation

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G-equivalence

Sign-inversion is a linear transformation S_i:

$$S_i(\alpha_j) = \alpha_j, j \neq i,$$
$$S_i(\alpha_i) = -\alpha_i.$$

q and q' are $\mathbb{Z}$-equivalent if one comes from another after a $\mathbb{Z}$-invertible linear transformation. For two $\mathbb{Z}$-equivalent forms q and q', q is non-negative (positive) iff q' is non-negative (positive).

q and q' are G-equivalent if one comes from another after a sequence of Gabrielov transformations, sign-inversions or a permutation of the variables.

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Associated bigraph

To an integer quadratic form q we associate bigraph B_q with vertices $1, \ldots, n$,
two vertices $i \neq j$ are jointed by $\max\{|A_{ij}|, |A_{ji}|\}$
full edges if $A_{ij} < 0$
dotted edges if $A_{ij} > 0$.
Edge starts at point with greater $|q_i|$.

Analoge of Ovsienko's Theorem

Theorem

Let q be connected positive quadratic form. Then there exists a finite sequence R of reductions such that R(q) is a connected classical positive form of Dynkin type $(A_n, D_n, B_n, C_n, G_2, F_4, E_6, E_7, E_8)$.

Theorem

If q is non-negative (positive) semi-integer form, then there is a sequence of inflation and deflations with composition T such that the bigraph of T(q) is disjoint union of unit Dynkin diagrams $(A_n, D_n, E_6, E_7, E_8)$ *multiplied by some non-negative (positive) integer.*

Quasi-Cartan matrix of quadratic form

A square matrix with integer coefficients C is called a quasi-Cartan matrix if it is symmetrizable (there exists a diagonal matrix D with positive diagonal entries such that DC is symmetric) and $C_{ii} = 2$ for all i.

A quasi-Cartan matrix is called Cartan matrix if it is positive definite and $C_{ij} \leq 0$ for all $i \neq j$.

For positive integer form $A_{ii} = 2$ and $A_{ij} = \frac{q_{ij}}{q_i}$ for $i \neq j$, and A_q is symmetrizable by matrix $D = \text{diag}(q_1, \ldots, q_n) \Rightarrow A_q$ is quasi-Cartan matrix

A_q is Cartan matrix iff form q is positive definite and classic.

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Lie algebra

Given a positive integer form q with quasi-Cartan matrix A let $\mathfrak{g}(q)$ be the Lie algebra defined by the generators $\{e_i, e_{-i}, h_i\}_{i\in\overline{1,n}}$ and the relations:

(w.1) $[h_i, h_j] = 0$;

(w.2) $[e_i, e_{-i}] = h_i$, $[e_i, e_{-j}] = 0$ for $i \neq j$;

(w.3) $[h_i, e_j] = A_{ij}e_j$, $[h_i, e_{-j}] = -A_{ij}e_{-j}$;

(θ_{ij}^{+}) $\mathrm{ad}(e_i)^{|A_{ij}|+1}(e_j) = 0, i \neq j$

(θ_{ij}^{-}) $\mathrm{ad}(e_{-i})^{|A_{ij}|+1}(e_{-j}) = 0, i \neq j$

Lie algebra of a classic positive integer form.

Theorem (Serre, [1])

If q is positive definite and classic integer form then $\mathfrak{g}(q)$ is a semisimple (and finite dimensional) Lie algebra.

Lie algebra associated to a non-negative unit form

Theorem (Barot, [2])

Two connected, non-negative unit forms q and q' are $\mathbb{Z}$-equivalent if and only if they are G-equivalent.

Theorem (Barot, [2])

If q and q' are G-equivalent then $\mathfrak{g}(q)$ and $\mathfrak{g}(q')$ are isomorphic as graded Lie algebras.

Equivalence

Theorem

Two connected, positive integer forms q and q' are $\mathbb{Z}$-equivalent and define identical sets of roots if and only if they are G-equivalent.

$x_1^2 - 3x_1x_2 + 3x_2^2$ and $x_1^2 - x_2^2$
are $\mathbb{Z}$-equivalent, but not G-equivalent.

Lie algebra associated to a positive integer quadratic form

Theorem

If q and q' are G-equivalent, then $\mathfrak{g}(q)$ and $\mathfrak{g}(q')$ are isomorphic as graded Lie algebras.

Theorem

Let q be a connected positive integer form and Δ its Dynkin type, then algebras $\mathfrak{g}(q) \simeq \mathfrak{g}(q_\Delta)$ are exactly finite-dimensional semisimple Lie algebras.

Summary

- Every positive integer form has corresponding uniquely defined Dynkin type.
- Every positive integer form defines Lie algebra in terms of the positive quasi-Cartan matrix.
- Associated Lie algebra is isomorphic to finite-dimensional semisimple Lie algebra of Dynkin type.

- Outlook
  - Properties of Lie algebra associated to non-negative integer quadratic form
    (non-negative quasi-Cartan matrix).
  - Properties of Lie algebra associated to any integer quadratic form
    (any quasi-Cartan matrix).

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