Teorie spolehlivosti

Priklad porovnani serioveho a paralelniho systemu

X_1, ..., X_n iid s rozdelenim exponencialnim s parametrem θ=1/λ, tj. s hustotou

```
Clear[pexp, θ];
d1 = ExponentialDistribution[1/θ];
pexp[x_] := PDF[d1, x]
pexp[x]
```

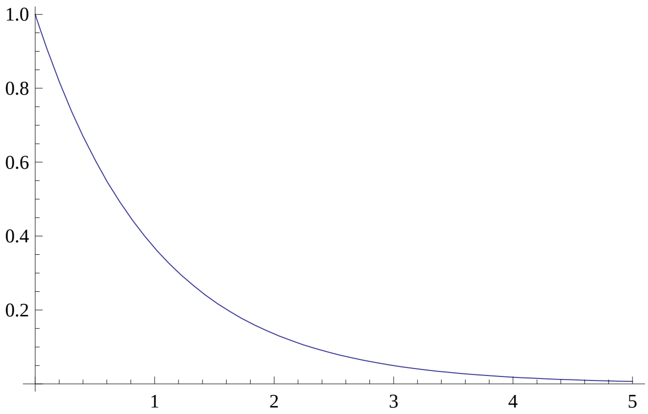
$$\frac{e^{-\frac{x}{\theta}}}{\theta}$$

distribucni funkci a funkci spolehlivosti

```
Clear[rexp];
CDF[d1, x]
rexp = 1 - CDF[d1, x]
Plot[rexp /. θ → 1, {x, 0, 5}]
```

$$\begin{cases} 1 - e^{-\frac{x}{\theta}} & x > 0 \end{cases}$$

$$1 - \begin{cases} 1 - e^{-\frac{x}{\theta}} & x > 0 \end{cases}$$



Intensita poruch je

```
Clear[rexp];
rexp[x_] := pexp[x] / e^(-x/θ)
rexp[y] // Simplify
```

$$\frac{1}{\theta}$$

Stredni doba do poruchy kazdeho prvku je

```
Mean[d1]
```

$$\theta$$

Stredni doba do poruchy zalohovaneho systemu s nezatizenymi zalohami $E\,X^{(n)} = n\,\theta$.

Doba do poruchy zalohovaneho systemu se zatizenymi zalohami ma distribucni funkci

$P\{X_{max} < x\} = P\{\forall i = 1, \; \ldots n \; X_i < x\} = F^n(x) =$

```
Clear[dfFn];
dfFn[x_] := (1 - Exp[-x/θ])^n
```

Stredni hodnota $E\, X_{max}$ je tedy

```
Assuming[n ∈ Integers, ∫_0^∞ (1 - dfFn[x]) dx]
Expand[dfFn[x]]
```

$\left(1 - e^{-\frac{x}{\theta}}\right)^n$

```
(1 - (1 - e^(-x/θ))^n) /. (1 - e^(-x/θ)) → y
```

$1 - y^n$

```
Solve[y == (1 - e^(-x/θ)), x]
```

Solve::ifun : Inverse functions are being used by Solve, so
some solutions may not be found; use Reduce for complete solution information. ≫

```
D[x /. {{x → -θ Log[1 - y]}}, y]
```

$\left\{\frac{\theta}{1-y}\right\}$

```
EXmax = θ Integrate[(1 - y^n)/(1 - y), {y, 0, 1}, Assumptions → {n ∈ Integers && n > 0}]
```

θ HarmonicNumber[n]

```
Series[HarmonicNumber[n], {n, ∞, 1}] // Normal // FullSimplify
```

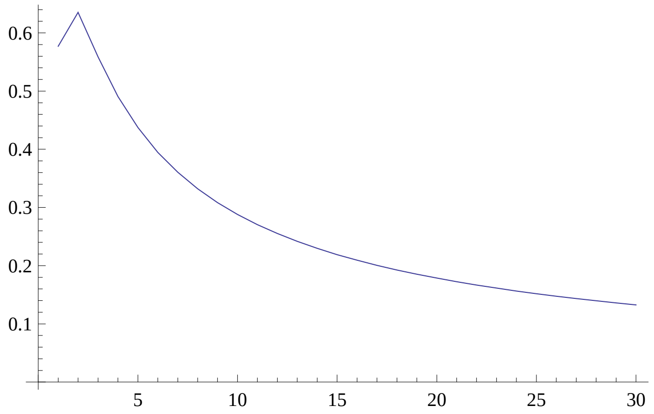
$\text{EulerGamma} + \frac{1}{2\,n} - \text{Log}\left[\frac{1}{n}\right]$

```
N[EulerGamma, 50]
```

0.57721566490153286060651209008240243104215933593992

Z toho vyplyva, ze Stredni hodnota $E\, X_{max}$ je priblizne $C + \log n$. Podil strednich hodnot je priblizne

```
ListPlot[Table[(Log[n] + EulerGamma)/n, {n, 30}], Joined → True, AxesOrigin → {0, 0}]
```



Pravdepodobnostni rozdeleni

Exponencialni rozdeleni Exp(θ)

```
CDF[d1, x] // First // First // First // FullForm
```

```
Plus[1, Times[-1, Power[E, Times[-1, x, Power[\[Theta], -1]]]]]
```

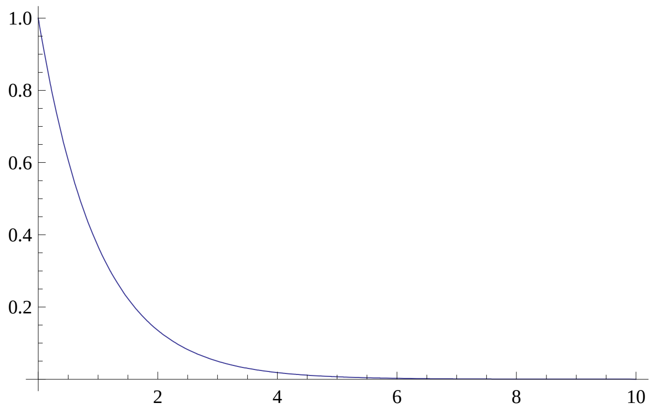
```
d1 = ExponentialDistribution[1/θ];
CDF[d1, x] // First // First // First
PDF[d1, x]
(* Intensita poruch *) PDF[d1, x] / (1 - (CDF[d1, x] // First // First // First))
```

$1 - e^{-\frac{x}{\theta}}$

$\frac{e^{-\frac{x}{\theta}}}{\theta}$

$\frac{1}{\theta}$

```
Plot[PDF[d1, x] /. θ → 1, {x, 0, 10}, PlotRange → All]
```



Momenty

```
ExpectedValue[x^k, d1, x, Assumptions → {k ∈ Integers && k > 0}]
```

θ^k Gamma[1 + k]

```
{Mean[d1], Variance[d1]}
```

$\{\theta, \theta^2\}$

```
{Skewness[d1], Kurtosis[d1]}
```

{2, 9}

Variacni koeficient $\sqrt{\text{var } X} \Big/ E\,X$

```
StandardDeviation[d1] / Mean[d1]
```

1

Rozdeleni N_t

```
PDF[GammaDistribution[p, a], y]
```

$$\frac{a^{-p}\, e^{-\frac{y}{a}}\, y^{-1+p}}{\text{Gamma}[p]}$$

```
CDF[GammaDistribution[p, a], y]
```

GammaRegularized$\left[p, 0, \frac{y}{a}\right]$

```
Assuming[k ∈ Integers && k > 0,
 CDF[GammaDistribution[k, 1/θ], t] - CDF[GammaDistribution[k + 1, 1/θ], t] // FullSimplify]
```

GammaRegularized[k, 0, t θ] - GammaRegularized[1 + k, 0, t θ]

```
(1/(θ^k Gamma[k]) Integrate[x^(k-1) Exp[-x/θ], {x, 0, t}, Assumptions → {k ∈ Integers && k > 0}] -
   1/(θ^(k+1) Gamma[k + 1]) Integrate[x^k Exp[-x/θ], {x, 0, t},
     Assumptions → {k ∈ Integers && k > 0}]) // FullSimplify
```

$$\frac{e^{-\frac{t}{\theta}} \left(\frac{t}{\theta}\right)^k}{\text{Gamma}[1+k]}$$

Weibullovo rozdeleni W(θ, β)

```
CDF[d2 = WeibullDistribution[β, θ], x]
```

$1 - e^{-\left(\frac{x}{\theta}\right)^{\beta}}$

```
Assuming[k ∈ Integers && k > 0, ExpectedValue[x^k, d2, x]]
```

$\theta^k \text{ Gamma}\left[\frac{k+\beta}{\beta}\right]$

```
{Mean[d2], Variance[d2]}
```

$\left\{\theta \text{ Gamma}\left[1+\frac{1}{\beta}\right],\ \theta^2\left(-\text{Gamma}\left[1+\frac{1}{\beta}\right]^2+\text{Gamma}\left[1+\frac{2}{\beta}\right]\right)\right\}$

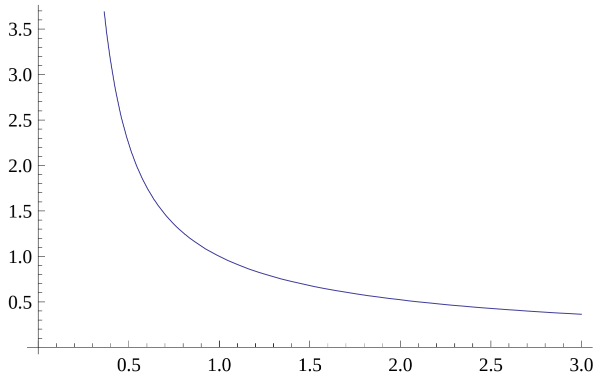
Variacni koeficient

```
vk = StandardDeviation[d2] / Mean[d2] // Simplify
```

$$\frac{\sqrt{-\text{Gamma}\left[1+\frac{1}{\beta}\right]^2+\text{Gamma}\left[\frac{2+\beta}{\beta}\right]}}{\text{Gamma}\left[1+\frac{1}{\beta}\right]}$$

nezavisi na θ !!!!

```
Plot[vk, {β, 0.1, 3}, AxesOrigin → {0, 0}]
```



Priklad (Odhad parametru β momentovou metodou)

```
Mean[WeibullDistribution[1.5, 1]]
randomsample = RandomReal[WeibullDistribution[1.5, 1], 50];
ex = Mean[randomsample]
sd = StandardDeviation[randomsample]
```

$$\text{fr = FindRoot}\left[\frac{\sqrt{-\text{Gamma}\left[1+\frac{1}{\beta}\right]^2+\text{Gamma}\left[\frac{2+\beta}{\beta}\right]}}{\text{Gamma}\left[1+\frac{1}{\beta}\right]} == \frac{\text{sd}}{\text{ex}},\ \{\beta, 1\}\right]$$

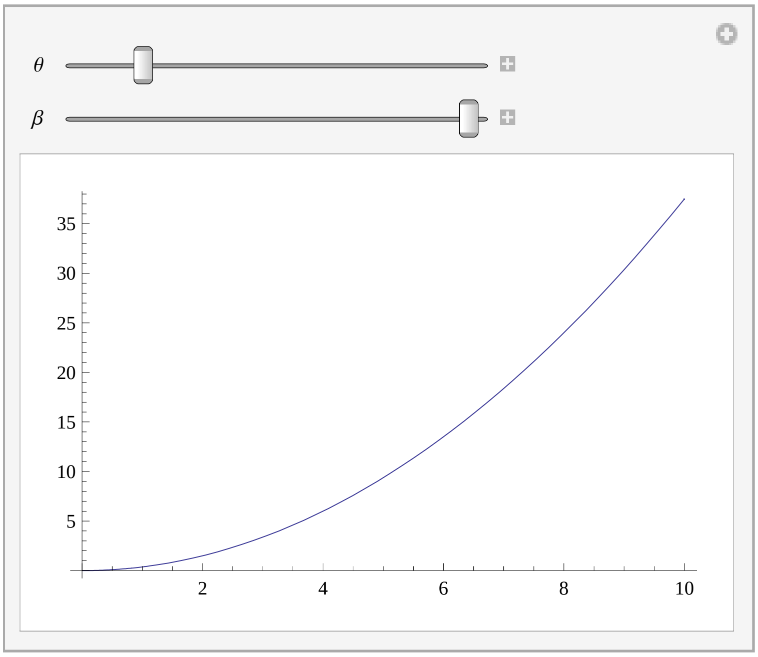
$$\text{odhad}\theta = \frac{\text{Gamma}\left[1+\frac{1}{\beta}\right]}{\text{ex}} \text{ /. fr}$$

```
0.902745

0.867631

0.509233

{β → 1.75927}

1.02618
```

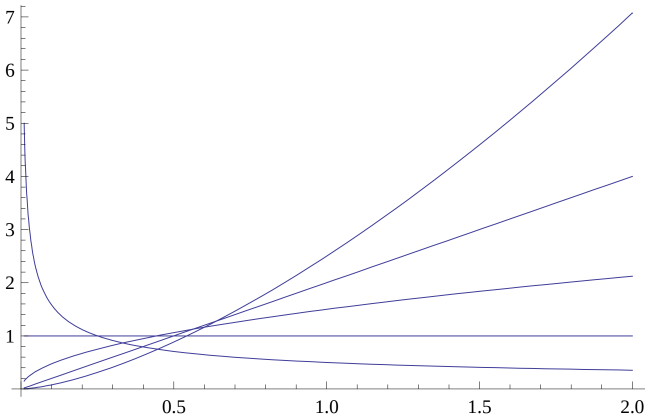
β /. fr

1.35525

$$\textbf{Manipulate}\left[\textbf{Plot}\left[\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}, \{x, 0.01, 10\}\right], \{\{\theta, 1\}, 0.5, 10\}, \{\{\beta, 1.5\}, 0.3, 3\}\right]$$



$$\textbf{Plot}\left[\textbf{Table}\left[\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \text{ /. } \theta \to 1, \{\beta, \{0.5, 1, 1.5, 2, 2.5\}\}\right], \{x, 0.01, 2\}\right]$$



Rayleighovo rozdeleni

$$\left\{\textbf{mr} = \textbf{Mean}\left[\textbf{d2} \text{ /. } \left\{\beta \to 2, \theta \to \sigma\sqrt{2}\right\}\right],\right.$$

$$\left.\textbf{sdr} = \textbf{StandardDeviation}\left[\textbf{d2} \text{ /. } \left\{\beta \to 2, \theta \to \sigma\sqrt{2}\right\}\right], \frac{\textbf{sdr}}{\textbf{mr}} \text{ // Simplify}, \frac{\textbf{sdr}}{\textbf{mr}} \text{ // N}\right\}$$

$$\left\{\sqrt{\frac{\pi}{2}}\,\sigma, \sqrt{2\left(1-\frac{\pi}{4}\right)}\,\sigma, \sqrt{-1+\frac{4}{\pi}}, 0.522723\right\}$$

Normalni rozdeleni $N(\mu, \sigma^2)$

```
d3 = NormalDistribution[μ, σ]
```

```
NormalDistribution[μ, σ]
```

Pst, ze nah. vel. dude mensi nez $\mu - 3\sigma$?

```
{CDF[d3, μ - 3 σ], CDF[d3, μ - 3 σ] // N}
```

$\left\{\frac{1}{2}\operatorname{Erfc}\left[\frac{3}{\sqrt{2}}\right], 0.0013499\right\}$

Pravidlo 2σ : $P(|\ N(\mu,\sigma) - \mu| < 2\sigma)$

```
CDF[d3, μ + 2 σ] - CDF[d3, μ - 2 σ] // N
```

```
0.9545
```

Pravidlo 3σ : $P(|\ N(\mu,\sigma) - \mu| < 3\sigma)$

```
CDF[d3, μ + 3 σ] - CDF[d3, μ - 3 σ] // N
```

```
0.9973
```

Hustota useknuteho normalniho rozdeleni

```
Clear[fr];
ftruncated[x_, μ_, σ_] :=
  1 / (σ CDF[NormalDistribution[0, 1], μ / σ]) PDF[NormalDistribution[0, 1], (x - μ) / σ]
```

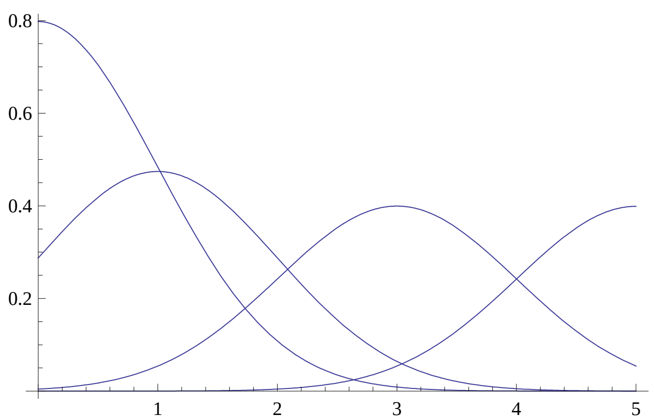
```
(Integrate[ftruncated[x, μ, σ], {x, 0, ∞}] /. If[a_, b_, c___] → b) // FullSimplify
```

$$\frac{\sqrt{\frac{1}{\sigma^2}}\ \sigma + \operatorname{Erf}\left[\frac{\mu}{\sqrt{2}\ \sigma}\right]}{1 + \operatorname{Erf}\left[\frac{\mu}{\sqrt{2}\ \sigma}\right]}$$

```
Integrate[ftruncated[x, μ, σ], {x, 0, ∞}, Assumptions → σ > 0]
```

```
1
```

```
Plot[Table[ftruncated[x, μ, 1], {μ, {0, 1, 3, 5}}], {x, 0, 5}]
```



Stredni hodnota

```
meantruncated = Integrate[x ftruncated[x, μ, σ], {x, 0, ∞}, Assumptions → σ > 0] // Simplify
```

$$\mu + \frac{e^{-\frac{\mu^2}{2\sigma^2}} \sqrt{\frac{2}{\pi}}\, \sigma}{1 + \mathrm{Erf}\left[\frac{\mu}{\sqrt{2}\,\sigma}\right]}$$

```
Table[{mm, "     " meantruncated /. {μ → mm, σ → 1}} // N, {mm, 0.5, 5, 0.5}] // TableForm
```

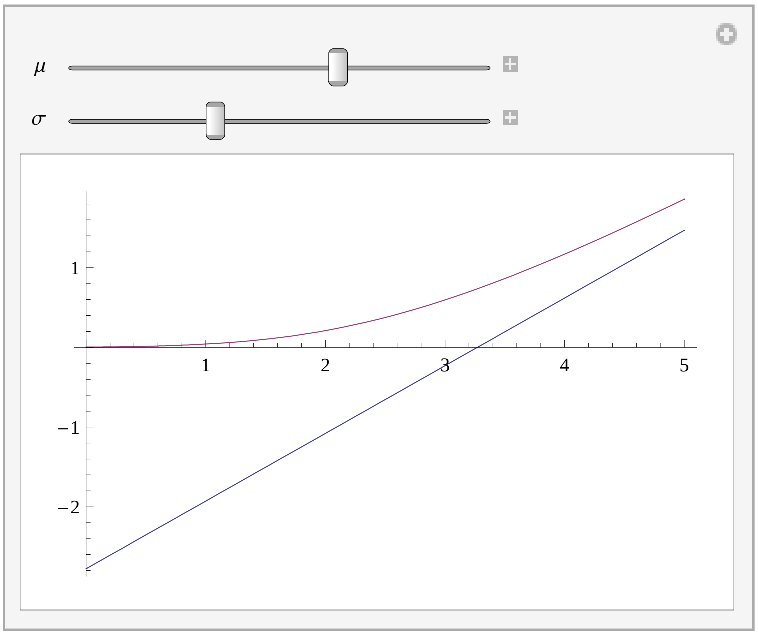
0.5	1.00916
1.	1.2876
1.5	1.63879
2.	2.05525
2.5	2.51764
3.	3.00444
3.5	3.50087
4.	4.00013
4.5	4.50002
5.	5.

Intensita poruch $N(\mu, \sigma)$

```
PDF[NormalDistribution[μ, σ], x]
```

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

```
Manipulate[
 Plot[{(x - μ)/σ^2, PDF[NormalDistribution[μ, σ], x] / (1 - CDF[NormalDistribution[μ, σ], x])},
  {x, 0, 5}], {μ, 0, 5}, {σ, 0.1, 3}]
```




```
PDF[NormalDistribution[μ, σ], x] /. {μ → 1, σ → 0.5, x → 5}
```

1.01045×10^{-14}

```
(1 - CDF[NormalDistribution[μ, σ], x]) /. {μ → 1, σ → 0.5, x → 5}
```

6.66134×10^{-16}

```
ser1 = Assuming[σ > 0, Series[PDF[NormalDistribution[μ, σ], x] /
    (1 - CDF[NormalDistribution[μ, σ], x]), {x, ∞, 3}]] // Simplify
```

$$\frac{x}{\sigma^2}-\frac{\mu}{\sigma^2}+\frac{1}{x}+\frac{\mu}{x^2}+\frac{\mu^2-2\sigma^2}{x^3}+O\left[\frac{1}{x}\right]^4$$

Priklad (prvek, poruchy nahle i postupne → seriovy system) se spolehlivosti

```
rR[x_, θ_, μ_, σ_] := Exp[-x/θ] CDF[NormalDistribution[μ, σ], x]

expected = Assuming[θ > 0 && σ > 0, ∫_0^∞ rR[x, θ, μ, σ] dx]
```

$$\frac{1}{2}\theta\left(e^{\frac{-2\theta\mu+\sigma^2}{2\theta^2}}\left(1+\mathrm{Erf}\left[\frac{\theta\mu-\sigma^2}{\sqrt{2}\,\theta\sigma}\right]\right)+\mathrm{Erfc}\left[\frac{\mu}{\sqrt{2}\,\sigma}\right]\right)$$

Log-normalni rozdeleni LN(μ, σ^2)

In[7]:=
```
PDF[d4 = LogNormalDistribution[μ, σ], x]
```

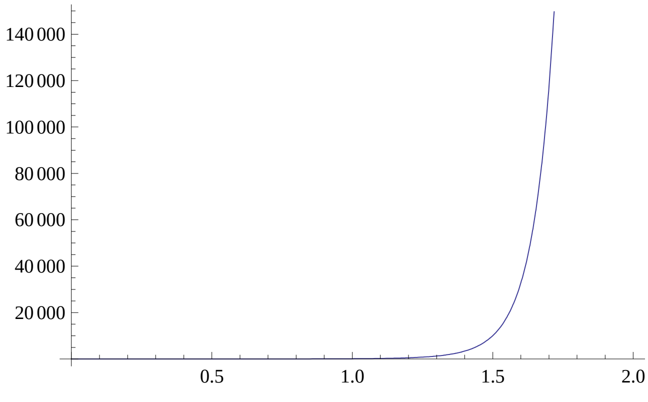
Out[7]=
$$\frac{e^{-\frac{(-\mu+\mathrm{Log}[x])^2}{2\sigma^2}}}{\sqrt{2\pi}\,x\,\sigma}$$

Momenty

```
{Mean[d4], Variance[d4], Skewness[d4], Kurtosis[d4]}
```

$$\left\{e^{\mu+\frac{\sigma^2}{2}},\ e^{2\mu+\sigma^2}\left(-1+e^{\sigma^2}\right),\ \sqrt{-1+e^{\sigma^2}}\left(2+e^{\sigma^2}\right),\ -3+3e^{2\sigma^2}+2e^{3\sigma^2}+e^{4\sigma^2}\right\}$$

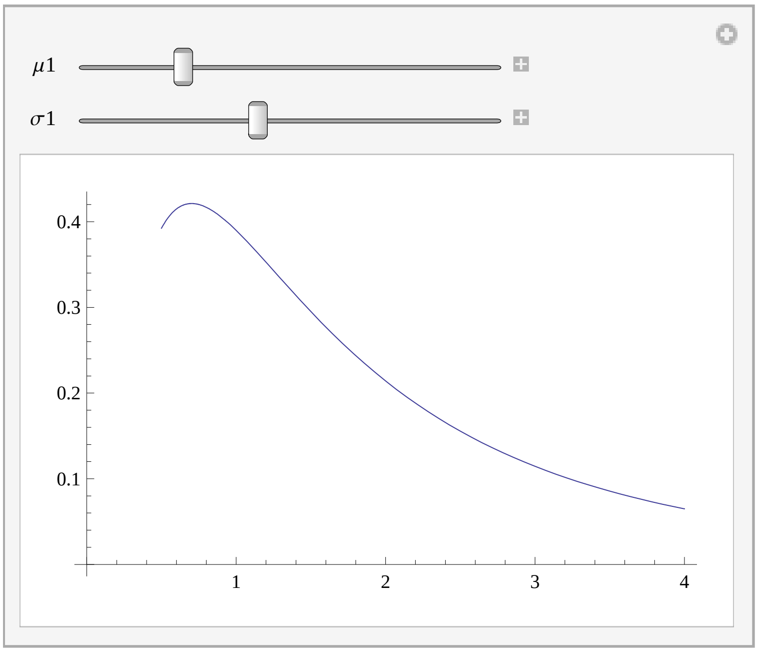
```
Plot[-3 + 3 e^(2 σ^2) + 2 e^(3 σ^2) + e^(4 σ^2), {σ, 0, 2}]
```




```
PDF[d4, x]
```

$$\frac{e^{-\frac{(-\mu+\mathrm{Log}[x])^2}{2\sigma^2}}}{\sqrt{2\pi}\,x\,\sigma}$$

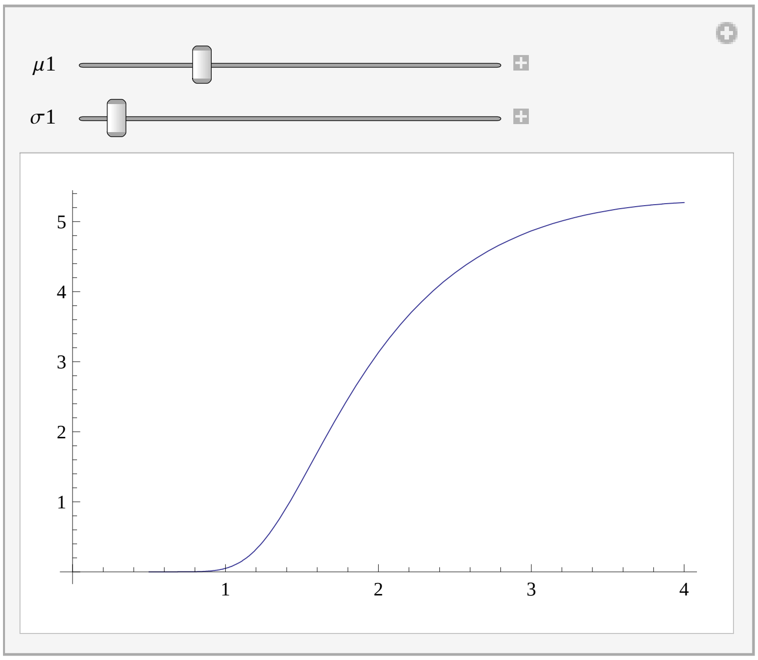
```
Manipulate[Plot[PDF[d4, x] /. {μ → μ1, σ → σ1}, {x, 0.5, 4},
  PlotRange → All, AxesOrigin → {0, 0}], {μ1, 0, 2}, {σ1, 0.1, 2}]
```



Intensita poruch

In[8]:=
```
Manipulate[Plot[PDF[d4, x] / (1 - CDF[d4, x]) /. {μ → μ1, σ → σ1},
  {x, 0.5, 4}, PlotRange → All, AxesOrigin → {0, 0}], {μ1, 0, 2}, {σ1, 0.1, 2}]
```

Out[8]=



Gamma rozdeleni Gamma(*a*, *p*)

```
PDF[d5 = GammaDistribution[p, 1 / a], x]
```

$$\frac{\left(\frac{1}{a}\right)^{-p} e^{-a x} x^{-1+p}}{\text{Gamma}[p]}$$

```
{Mean[d5], Variance[d5], Skewness[d5], Kurtosis[d5]}
```

$$\left\{a\,p,\ a^2\,p,\ \frac{2}{\sqrt{p}},\ 3+\frac{6}{p}\right\}$$

$$\frac{\int_x^\infty t^{p-1}\,e^{-a\,t}\,dt}{x^{p-1}\,e^{-a\,x}}$$

$$e^{a\,x}\,x^{1-p}\ \text{If}\left[\text{Re}[a]>0\ \&\&\ (\text{Im}[x]\neq 0\ ||\ x>0),\ a^{-p}\,\text{Gamma}[p]+x^p\,(a\,x)^{-p}\,(-\text{Gamma}[p]+\text{Gamma}[p,a\,x]),\ \text{Integrate}\left[e^{-a\,t}\,t^{-1+p},\ \{t,x,\infty\},\ \text{Assumptions}\to x\leq 0\ ||\ \text{Re}[a]\leq 0\right]\right]$$

$$\text{expr}=\text{Assuming}\left[x>0\ \&\&\ p>0\ \&\&\ a>0,\ \frac{\int_x^\infty t^{p-1}\,e^{-a\,t}\,dt}{x^{p-1}\,e^{-a\,x}}\right]$$

$$a^{-p}\,e^{a\,x}\,x^{1-p}\,\text{Gamma}[p,a\,x]$$

```
dexpr = D[expr, x] // FullSimplify
```

$$a^{-p}\,x^{-p}\,\left(-(a\,x)^p+e^{a\,x}\,(1-p+a\,x)\,\text{Gamma}[p,a\,x]\right)$$

```
Reduce[dexpr < 0 && a > 0 && p > 0, {x, a, p}, Reals]
```

Reduce::nsmet : This system cannot be solved with the methods available to Reduce. ≫

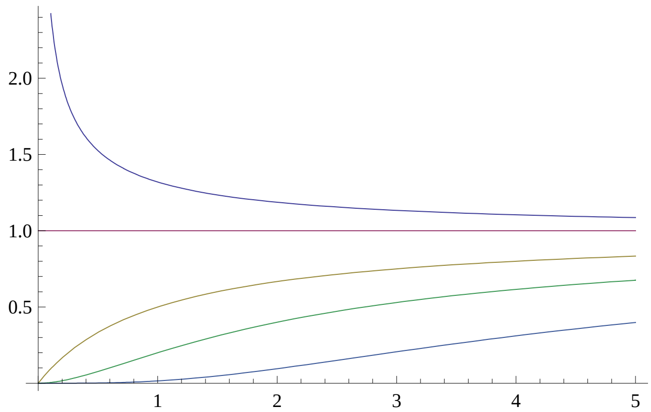
$$\text{Reduce}\left[a^{-p}\,x^{-p}\,\left(-(a\,x)^p+e^{a\,x}\,(1-p+a\,x)\,\text{Gamma}[p,a\,x]\right)<0\ \&\&\ a>0\ \&\&\ p>0,\ \{x,a,p\},\ \text{Reals}\right]$$

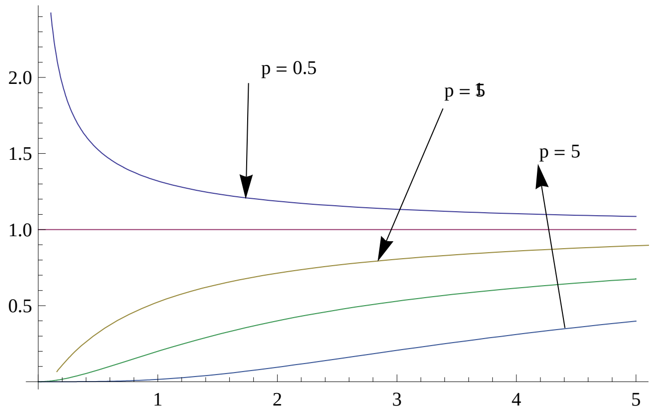
Grafy intensity poruch

$$\text{Table}\left[\frac{x^{p-1}\,e^{-x}}{\text{Assuming}\left[x>0,\ \int_x^\infty t^{p-1}\,e^{-t}\,dt\right]},\ \{p,\{0.5,1,2,3,5\}\}\right]\ //\ \text{Evaluate}$$

$$\left\{\frac{e^{-x}}{x^{0.5}\left(1.77245-2.\,x^{0.5}\,\text{Hypergeometric1F1}[0.5,1.5,-x]\right)},\ 1,\ \frac{x}{1+x},\ \frac{x^2}{2+x\,(2+x)},\ \frac{x^4}{24+x\,(24+x\,(12+x\,(4+x)))}\right\}$$

$$\text{Plot}\left[\text{Table}\left[\frac{x^{p-1}\,e^{-x}}{\text{Assuming}\left[x>0,\ \int_x^\infty t^{p-1}\,e^{-t}\,dt\right]},\ \{p,\{0.5,1,2,3,5\}\}\right]\ //\ \text{Evaluate},\ \{x,0.001,5\}\right]$$





$$\text{Plot}\left[\frac{x^{p-1}\, e^{-x}}{\int_x^\infty t^{p-1}\, e^{-t}\, dt} \;/.\; p \to 0.5,\ \{x, 0.001, 5\}\right]$$